Case studies for the multilinear Kakeya theorem and Wolff-type inequalities

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Declaration

I declare that this thesis was composed by myself and that the work it contains is my own, except where explicitly stated otherwise in the text. The work has not been submitted for any other degree or professional qualification.

George Kinnear

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Abstract

This thesis is concerned with two different problems in harmonic analysis: the multilinear Kakeya theorem, and Wolff-type inequalities for paraboloids. Chapter 1 gives an overview of both of these problems.

In Chapter 2 we investigate an important special case of the multilinear Kakeya theorem, the so-called "bush example". While the endpoint case of the multilinear Kakeya theorem was recently proved by Guth, the proof is highly abstract; our aim is to provide a more elementary proof in this special case. This is achieved for a significant part of the three-dimensional case in the main result of the chapter.

Chapter 3 is a study of the endpoint case of a mixed-norm Wolff-type inequality for the paraboloid. The main result adapts an example of Bourgain to show that the endpoint inequality cannot hold with an absolute constant; there must be a dependence on the thickening of the paraboloid. The remainder of the chapter is a series of case studies, through which we establish positive endpoint results for certain classes of function, as well as indicating specific examples which need to be better understood in order to obtain the full endpoint result.

Lay summary

This thesis looks at two different problems in the area of harmonic analysis.

The first is the multilinear Kakeya problem, which involves taking a collection of tubes and measuring the space inside them in two different ways — the problem is to show that no matter what collections of tubes are used, one measurement is always smaller than the other. This result was proved only very recently; originally by Guth in 2010, with improvements by Bourgain and Guth in 2011, and Carbery and Valdimarsson in 2012. These proofs are very abstract, involving sophisticated geometric ideas, so our aim is to give a simpler proof. To make this possible, we only consider certain collections of tubes — ones where all the tubes go through the same point, so we have a sort of "bush" of tubes. Imposing this extra condition does make the problem simpler, but knowing how to deal with the bush example is a good first step to understanding the whole problem. The main result in Chapter 2 is that if we take the bush example in three-dimensional space and measure the tubes in a certain way (not quite the same as in the original question) then it is indeed always less than the other measurement.

The second problem we consider is a Wolff-type inequality, which has a similar goal of showing that one way of measuring an object always gives a smaller answer than another. The problem involves working in a region which has a certain "thickness"; the main result in Chapter 3 is that the statement "one measurement is less than the other" can only be true if we include some dependence on this thickness. Showing that the statement holds for all possible objects is very difficult, so we consider some important examples and show that the result holds for these.

List of notation

$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathbb{S}^n$	the sphere of radius 1 in $\mathbb{R}^n$
σ	surface measure on the sphere
χ	characteristic function; $\chi_E(x)$ is 1 if $x \in E$ and 0 otherwise
$B_r(x)$	the ball of radius r centred at $x \in \mathbb{R}^n$
$A \lesssim B$	$A \leq CB$ for some constant C
$A \gtrsim B$	$A \geq CB$ for some constant C
$A \sim B$	$A \lesssim B$ and $A \gtrsim B$
$f(x) = O(g(x))$	$f(x) \lesssim g(x)$ for sufficiently large (or small, depending on context) values of x
$\|A\|$	the (Lebesgue) measure of the set A
$\#A$	the number of elements in the finite set A
$\\|\cdot\\|_p$	the L^p norm
$\mathcal{S}(\mathbb{R}^n)$	the Schwartz space of smooth, rapidly decreasing functions on $\mathbb{R}^n$
$\lg x$	$\log_2 x$

CHAPTER

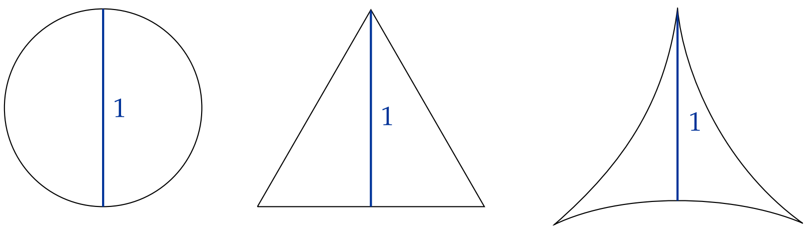
1 Introduction

1.1 The Kakeya problem

We begin with an introduction to the problem which is studied in Chapter 2.

Definition 1.1. A *Kakeya set* is a compact set in $\mathbb{R}^n$ containing a unit line segment in every direction $\omega \in \mathbb{S}^{n-1}$. $\Diamond$

Clearly $B_{1/2}(0) = \left\{x \in \mathbb{R}^n : |x| \leq \frac{1}{2}\right\}$ is a Kakeya set, but this example does not have the smallest possible volume. For instance, in $\mathbb{R}^2$ the circle of area $\frac{\pi}{4}$ can be replaced with an equilateral triangle of area $\frac{1}{\sqrt{3}}$, or even a deltoid of area $\frac{\pi}{8}$.



In fact, Besicovitch showed that a Kakeya set can have Lebesgue measure zero [Bes28]. This leads naturally to the question of how large such sets must be in other senses; for instance, in terms of their Hausdorff dimension.

Definition 1.2. For each $\alpha > 0$, let

$$m_\alpha(E) = \lim_{\varepsilon \to 0} \inf \left\{ \sum_{i=1}^{\infty} r(B_i)^\alpha \ : \ E \subseteq \cup_{i=1}^{\infty} B_i \text{ and } r(B_i) \leq \varepsilon \right\},$$

where each $B_i \subset \mathbb{R}^n$ is a ball of radius $r(B)$, and the infimum is over all possible choices of the B_i.

Then the *Hausdorff dimension* of E is

$$\dim_{\mathcal{H}}(E) = \inf\{\alpha : m_\alpha(E) = 0\}. \qquad \Diamond$$

It is thought that Kakeya sets must have full Hausdorff dimension:

Conjecture 1.3 (Kakeya conjecture)**.** *For any Kakeya set $E \subseteq \mathbb{R}^n$, $\dim_{\mathcal{H}}(E) = n$.*

This has been established for $n = 2$ [Dav71], but remains unproven for $n \geq 3$.

1.1.1 Maximal function formulation

Definition 1.4. Given a locally integrable function f defined on $\mathbb{R}^n$, the *Kakeya maximal function $M_N f$* is defined by

$$(M_N f)(\omega) = \sup_T \frac{1}{|T|} \int_T |f(x)|\, dx,$$

where the supremum is over $1 \times \cdots \times 1 \times N$ tubes T in the direction ω; i.e. cylinders in $\mathbb{R}^n$ with diameter 1 and long dimension N, with the long dimension parallel to $\omega \in \mathbb{S}^{n-1}$. $\Diamond$

Conjecture 1.5 (Kakeya maximal function conjecture)**.**

$$\|M_N f\|_{L^n(\mathbb{S}^{n-1})} \leq C N^{-1} (\log N)^{\frac{n-1}{n}} \|f\|_{L^n(\mathbb{R}^n)}. \tag{1.1}$$

It can be shown that Conjecture 1.5 implies Conjecture 1.3 (e.g. in [Tao99, Lecture 5, Proposition 2.3]).

Note that (1.1) can be restated in the dual form

$$\int_{\mathbb{R}^n} \left(\sum_{T \in \mathbf{T}} c_T \chi_T(x) \right)^{\frac{n}{n-1}} dx \leq C_n N \log N \sum_{T \in \mathbf{T}} c_T^{\frac{n}{n-1}} \tag{1.2}$$

where the $\mathbf{T}$ is a set of $1 \times \cdots \times 1 \times N$ tubes whose directions $e(T)$ are $\frac{1}{N}$-separated on $\mathbb{S}^{n-1}$ (i.e. for distinct tubes $T, T' \in \mathbf{T}$, we have $|e(T) - e(T')|_{\mathbb{S}^{n-1}} \geq \frac{1}{N}$), and $c_T \geq 0$.

Establishing (1.2) is a difficult open problem, except when $n = 2$. There is partial progress for $n \geq 3$; see for instance [Bou91], [Wol95], [TVV98], [KT02].

1.1.2 The bush example

A key first step — and a central idea introduced in [Bou91] — is to consider the bush example, where all the tubes pass through a common point.

Observation 1.6 (Bush example)**.** *If we impose the additional condition that all the tubes pass through the origin, then* (1.2) *is true.*

Proof. Using polar coordinates, we can write the left-hand side of (1.2) as

$$\begin{aligned}
&\int_{r=0}^{N}\int_{\mathbb{S}^{n-1}}\left(\sum_{T\in\mathbf{T}} c_T\chi_T(ru)\right)^{\frac{n}{n-1}} r^{n-1}\,d\sigma(u)\,dr\\
&\lesssim \sum_{k=1}^{\log N}\int_{r\sim 2^k} r^{n-1}\,dr\int_{u\in\mathbb{S}^{n-1}}\left(\sum_{T\in\mathbf{T}} c_T\chi_T(2^k u)\right)^{\frac{n}{n-1}} d\sigma(u)\\
&\lesssim \sum_{k=1}^{\log N} 2^{kn}\int_{u\in\mathbb{S}^{n-1}}\left(\sum_{T\in\mathbf{T}} c_T\chi_T(2^k u)\right)^{\frac{n}{n-1}} d\sigma(u)
\end{aligned}$$

Now for fixed k, we take a set of 2^{-k}-separated points $\omega_\ell \in \mathbb{S}^{n-1}$ and form a finitely overlapping covering of $\mathbb{S}^{n-1}$ by the "caps"

$$\mathrm{cap}_{2^{-k}}(\omega_\ell) = \left\{u\in\mathbb{S}^{n-1} \,:\, |u-\omega_\ell|_{\mathbb{S}^{n-1}} \le 2^{-k}\right\}.$$

Thus our bound on the left-hand side of (1.2) becomes

$$\sum_{k=1}^{\log N} 2^{kn}\sum_{\ell}\int_{u\in\mathrm{cap}_{2^{-k}}(\omega_\ell)}\left(\sum_{T\in\mathcal{T}_{k,\ell}} c_T\chi_T(2^k u)\right)^{\frac{n}{n-1}} d\sigma(u), \tag{1.3}$$

where $\mathcal{T}_{k,\ell}$ is the set of tubes which give a nonzero contribution. How large is $\mathcal{T}_{k,\ell}$?

- To have $\chi_T(2^k u) \neq 0$, we must have $\omega = e(T)$ within 2^{-k} of u, which itself is within 2^{-k} of ω_ℓ, so $|\omega-\omega_\ell|_{\mathbb{S}^{n-1}} \le 2\times 2^{-k}$, i.e. $\omega\in\mathrm{cap}_{2\times 2^{-k}}(\omega_\ell)$.
- The tube directions are $\frac{1}{N}$-separated on $\mathbb{S}^{n-1}$, so within $\mathrm{cap}_{2\times 2^{-k}}(\omega_\ell)$ the number of different tube directions is

$$\lesssim \frac{|\mathrm{cap}_{2\times 2^{-k}}(\omega_\ell)|}{|\mathrm{cap}_{1/N}|} \sim \frac{(2^{1-k})^{n-1}}{(1/N)^{n-1}} = N^{n-1}2^{(1-k)(n-1)}.$$

  Thus $\#\mathcal{T}_{k,\ell} \lesssim N^{n-1}2^{(1-k)(n-1)}$.

Now applying Hölder's inequality to (1.3), we obtain the bound

$$
\begin{aligned}
&\sum_{k=1}^{\log N} 2^{kn} \sum_{\ell} \int_{|u-\omega_\ell|\le 2^{-k}} \left(\sum_{T\in\mathcal{T}_{k,\ell}} \chi_T(2^k u)^n \right)^{\frac{n}{n-1}\frac{1}{n}} \left(\sum_{T\in\mathcal{T}_{k,\ell}} c_T^{\frac{n}{n-1}} \right)^{\frac{n}{n-1}\frac{n-1}{n}} d\sigma(u) \\
&\lesssim \sum_{k=1}^{\log N} 2^{kn} \sum_{\ell} |\mathrm{cap}_{2^{-k}}(\omega_\ell)| (\#\mathcal{T}_{k,\ell})^{\frac{1}{n-1}} \left(\sum_{T\in\mathcal{T}_{k,\ell}} c_T^{\frac{n}{n-1}} \right) \\
&\lesssim \sum_{k=1}^{\log N} 2^{kn} \sum_{\ell} (2^{-k})^{n-1} \left(N^{n-1} 2^{(1-k)(n-1)} \right)^{\frac{1}{n-1}} \sum_{T\in\mathcal{T}_{k,\ell}} c_T^{\frac{n}{n-1}} \\
&\lesssim \sum_{k=1}^{\log N} N \sum_{\ell} \sum_{T\in\mathcal{T}_{k,\ell}} c_T^{\frac{n}{n-1}} \\
&= N \log N \sum_{T\in\mathbf{T}} \sum_{\ell \text{ s.t. } \mathcal{T}_{k,\ell}\ni T} c_T^{\frac{n}{n-1}}
\end{aligned}
$$

The result then follows, since the covering of S^{n-1} indexed by ℓ is finitely overlapping, so there are $O(1)$ ℓs for each T. ■

1.1.3 The multilinear Kakeya problem

Note that in the dual form of the Kakeya maximal function conjecture, (1.2), we can write the left-hand side as

$$
\int_{\mathbb{R}^n} \left(\sum_{T\in\mathbf{T}} c_T \chi_T(x) \right)^{\frac{n}{n-1}} dx = \int_{\mathbb{R}^n} \prod_{j=1}^{n} \left(\sum_{T\in\mathbf{T}} c_T \chi_T(x) \right)^{\frac{1}{n-1}} dx.
$$

A contribution to this is

$$
\int_{\mathbb{R}^n} \prod_{j=1}^{n} \left(\sum_{T\in\mathbf{T}_j} c_T \chi_T(x) \right)^{\frac{1}{n-1}} dx,
$$

where the $\mathbf{T}_j \subseteq \mathbf{T}$ are certain families of tubes.

Definition 1.7 (Transverse families)**.** The families $\mathbf{T}_j$ are *transverse* if each $T \in \mathbf{T}_j$ has $e(T)$ in a small, fixed neighbourhood of the basis vector e_j. ♢

The following conjecture arose in [BCT06, Conjecture 1.8 and Remark 1.11].

Conjecture 1.8 (Multilinear Kakeya conjecture)**.** *If the families $\mathbf{T}_j$ are transverse, then there is a constant C_n independent of the families of tubes so that for any choice of nonnegative* *coefficients* c_T,

$$\int_{\mathbb{R}^n} \prod_{j=1}^{n} \left(\sum_{T \in \mathbf{T}_j} c_T \chi_T(x) \right)^{\frac{1}{n-1}} dx \leq C_n \prod_{j=1}^{n} \left(\sum_{T \in \mathbf{T}_j} c_T \right)^{\frac{1}{n-1}}. \tag{1.4}$$

This can be used to deal with the contribution in (1.2) due to transverse intersections of tubes, since applying Hölder's inequality followed by the inequality of arithmetic and geometric means gives

$$\begin{aligned} \prod_{j=1}^{n} \left(\sum_{T \in \mathbf{T}_j} c_T \right)^{\frac{1}{n-1}} &\leq \prod_{j=1}^{n} \left((\#\mathbf{T}_j)^{\frac{1}{n-1}} \sum_{T \in \mathbf{T}} c_T^{\frac{n}{n-1}} \right)^{\frac{1}{n}} \\ &\leq \frac{1}{n} \sum_{j=1}^{n} (\#\mathbf{T}_j)^{\frac{1}{n-1}} \sum_{T \in \mathbf{T}} c_T^{\frac{n}{n-1}}, \end{aligned}$$

and since the $\mathbf{T}_j \subseteq \mathbf{T}$ have $1/N$-separated directions,

$$\sum_{j=1}^{n} (\#\mathbf{T}_j)^{\frac{1}{n-1}} \lesssim \sum_{j=1}^{n} (N^{n-1})^{\frac{1}{n-1}} \lesssim N.$$

Thus we obtain the right-hand side of (1.2), without needing the factor of $\log N$ which appears there.

Remark 1.9. While we have used (1.4) to deal with a certain part of the linear problem (1.2), it is worth noting that (1.4) enjoys some gains over what is true in the linear case. For instance, it is known that (1.4) holds even if the tubes in each $\mathbf{T}_j$ can have the same direction (i.e. removing the assumption that the directions $e(T)$ are $1/N$-separated). We also have ℓ^1 norms in the right-hand side of (1.4), which are stronger than the $\ell^{n/(n-1)}$ norms appearing in (1.2). And perhaps most strikingly, there is no dependence on the parameter N in (1.4). $\Diamond$

Conjecture 1.8 arose in [BCT06] as (1.4) is the endpoint case of the multilinear Kakeya theorem — which was proved up to, but not including, the endpoint. The endpoint result was first proved in [Gut10], using algebraic topology.

Theorem 1.10 (Transverse tubes). *If the families* $\mathbf{T}_j$ *are transverse, we have*

$$\int_{\mathbb{R}^n} \left(\sum_{T_1 \in \mathbf{T}_1} c_{T_1} \chi_{T_1}(x) \cdots \sum_{T_n \in \mathbf{T}_n} c_{T_n} \chi_{T_n}(x) \right)^{\frac{1}{n-1}} dx \lesssim \prod_{j=1}^{n} \left(\sum_{T \in \mathbf{T}_j} c_T \right)^{\frac{1}{n-1}}. \tag{1.5}$$

In fact, Guth proved a more general statement, involving the quantity $\omega_1 \wedge \cdots \wedge \omega_n$.

Definition 1.11 ($\omega_1 \wedge \cdots \wedge \omega_n$)**.** Given $\omega_1, \ldots, \omega_n \in \mathbb{S}^{n-1}$, we define $\omega_1 \wedge \cdots \wedge \omega_n$ to be the volume of the parallelepiped in $\mathbb{R}^n$ with edges ω_i. This is also given by $|\det W|$, where W is the $n \times n$ matrix with columns ω_i. ◊

Theorem 1.12 (Quantitatively transverse tubes)**.** *If the families* $\mathbf{T}_j$ *are* quantitatively transverse, *i.e.* $e(T_1) \wedge \cdots \wedge e(T_n) \geq \alpha$ *for all* $T_j \in \mathbf{T}_j$*, then*

$$\int_{\mathbb{R}^n} \left(\sum_{T_1 \in \mathbf{T}_1} \chi_{T_1}(x) \cdots \sum_{T_n \in \mathbf{T}_n} \chi_{T_n}(x) \right)^{\frac{1}{n-1}} dx \lesssim \alpha^{-\frac{1}{n-1}} \left(\#\mathbf{T}_1 \ldots \#\mathbf{T}_n \right)^{\frac{1}{n-1}} . \tag{1.6}$$

A further generalisation of this was established in [BG11, §7] (although the result there is for curved tubes, which we shall not consider here).

Theorem 1.13 (Arbitrary tubes)**.** *For arbitrary families* $\mathbf{T}_j$*, we have*

$$\int_{\mathbb{R}^n} \left(\sum_{T_1 \in \mathbf{T}_1} \chi_{T_1}(x) \cdots \sum_{T_n \in \mathbf{T}_n} \chi_{T_n}(x) e(T_1) \wedge \cdots \wedge e(T_n) \right)^{\frac{1}{n-1}} dx \lesssim \left(\#\mathbf{T}_1 \ldots \#\mathbf{T}_n \right)^{\frac{1}{n-1}} . \tag{1.7}$$

This was reproved in [CV12], using the Borsuk-Ulam theorem in place of the more sophisticated algebraic topology employed by Bourgain and Guth.

Remark 1.14. Note that (1.7) $\Rightarrow$ (1.6) $\Rightarrow$ (1.5). ◊

In Chapter 2, we investigate the bush example in the multilinear setting and easily establish analogues of (1.5) and (1.6). The main result of Chapter 2 addresses the bush example in the "arbitrary tubes" case (1.7), and establishes the bound for the $n = 3$ case (at least for what is considered to be the main term).

1.2 Wolff-type inequalities

We now introduce the setting of the problem considered in Chapter 3.

In [Wol00], Wolff introduced a certain inequality involving L^p norms and a decomposition of the light cone $\left\{ \xi \in \mathbb{R}^{2+1} : \xi_3 = \sqrt{\xi_1^2 + \xi_2^2} \right\}$, and established it for $p > 74$. This was extended to higher dimensions in [ŁW02], and the cone was replaced with more general surfaces in [ŁP06].

The conjectured Wolff inequality for paraboloids states that for all $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$\Big\|\sum_j f_j\Big\|_p \leq C_\varepsilon \delta^{-\alpha(p)-\varepsilon}\left(\sum_j \|f_j\|_p^p\right)^{1/p}, \tag{1.8}$$

where $\alpha(p) = \frac{n-1}{2} - \frac{n}{p}$ is the standard Bochner-Riesz exponent (e.g. in [Car92]) and the f_j have supp $\hat{f}_j \subseteq S_j$ for some "slabs" S_j. Specifically, we take a δ-neighbourhood of the truncated paraboloid,

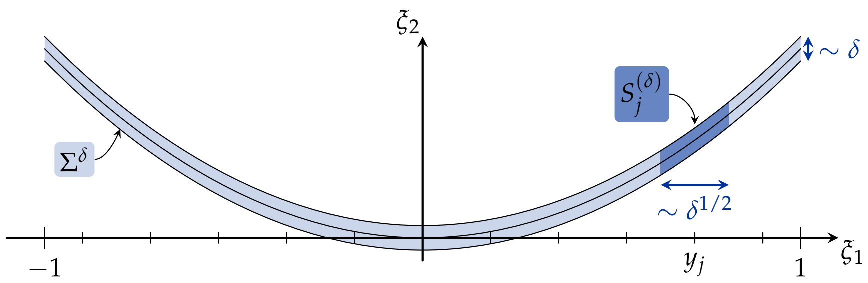
$$\Sigma^\delta = \left\{(\xi', \xi_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \left|\xi_n - \tfrac{1}{2}|\xi'|^2\right| \leq \delta, |\xi'| \leq 1\right\}$$

and decompose it into "slabs" having all other dimensions $\delta^{1/2}$. To do this, take a $\delta^{1/2}$-separated sequence $\{y_j\} \subset \mathbb{R}^{n-1}$ and form the slabs

$$S_j^{(\delta)} = \left\{(\xi', \xi_n) \in \Sigma^\delta : |\xi' - y_j| \leq C\delta^{1/2}\right\}.$$

Typically, we use $y_j = \delta^{1/2} j$ with $j \in \mathbb{Z}^{n-1}$.

This setup is illustrated for the $n = 2$ case in the following diagram:



Remark 1.15. The conjectured range of p for which (1.8) can hold is

$$p \geq 2 + \frac{4}{n-1} = 2\frac{n+1}{n-1}.$$

To see this, we use the n-dimensional Rademacher functions $r_j(t)$ defined on $[0,1]$ (see [Ste70, p104] for details). These have the key property that, with $Q = [0,1]^n$ the unit cube in $\mathbb{R}^n$,

$$\left(\int_Q \Big|\sum_j a_j r_j(t)\Big|^p dt\right)^{1/p} \sim \left(\sum_j |a_j|^2\right)^{1/2} \tag{1.9}$$

for every $p < \infty$ (whenever the right-hand side is finite), with the implied constant depending on p [Ste70, (44) on p104].

Now let $\widehat{h_j}(\xi) = \phi\left(\frac{\xi - w_j}{\delta}\right)$ with $\phi \in C^\infty$ supported in $|\xi| \leq \frac{1}{10}$, and $w_j = (y_j, \frac{1}{2}|y_j|^2)$, and put

$$h_{j,t}(x) = h_j(x) r_j(t)$$

for $t \in [0,1]$. Since the $h_{j,t}$ are of the form required of the f_j in Wolff's inequality, (1.8) would imply

$$\left(\int_Q \Big\|\sum_j h_{j,t}\Big\|_p^p \, dt\right)^{1/p} \leq C_\varepsilon \delta^{-\alpha(p)-\varepsilon} \left(\int_Q \sum_j \|h_{j,t}\|_p^p \, dt\right)^{1/p}.$$

Applying Fubini's theorem and (1.9) to the expression on the left-hand side,

$$\begin{aligned}
\int_Q \Big\|\sum_j h_{j,t}\Big\|_p^p \, dt &= \int_Q \int_{\mathbb{R}^n} \Big|\sum_j h_j(x) r_j(t)\Big|^p \, dx \, dt \\
&\sim \int_{\mathbb{R}^n} \left(\sum_j |h_j(x)|^2\right)^{p/2} dx,
\end{aligned}$$

while on the right-hand side we have

$$\begin{aligned}
\int_Q \sum_j \|h_{j,t}\|_p^p \, dt &= \int_Q \sum_j \int_{\mathbb{R}^n} |h_j(x) r_j(t)|^p \, dx \, dt \\
&= \sum_j \int_Q |r_j(t)|^p \, dt \int_{\mathbb{R}^n} |h_j(x)|^p \, dx \\
&\leq \sum_j \int_{\mathbb{R}^n} |h_j(x)|^p \, dx = \sum_j \|h_j\|_p^p,
\end{aligned}$$

since $|r_j(t)| \leq 1$.

Thus (1.8) implies

$$\left\|\Big(\sum_j |h_j|^2\Big)^{1/2}\right\|_p \lesssim C_\varepsilon \delta^{-\alpha(p)-\varepsilon} \left(\sum_j \|h_j\|_p^p\right)^{1/p}.$$

Now using $|h_j(x)| = \delta^n |\widehat{\phi}(\delta x)|$ we find

- $\left\|\Big(\sum_j |h_j|^2\Big)^{1/2}\right\|_p = \delta^n \left\|\Big(\sum_j |\widehat{\phi}(\delta \cdot)|^2\Big)^{1/2}\right\|_p = \delta^n (\#j\text{'s})^{1/2} \|\widehat{\phi}(\delta \cdot)\|_p$
- $\|h_j\|_p = \delta^n \|\widehat{\phi}(\delta \cdot)\|_p$

so overall (1.8) implies

$$\begin{aligned} \delta^n (\#j\text{'s})^{1/2} \left\| \widehat{\phi}(\delta \cdot) \right\|_p &\lesssim \delta^{-\alpha(p)-\varepsilon} \delta^n \left\| \widehat{\phi}(\delta \cdot) \right\|_p (\#j\text{'s})^{1/p} \\ \text{i.e.} \quad (\#j\text{'s})^{\frac{1}{2}-\frac{1}{p}} = \delta^{-\frac{n-1}{4}+\frac{n-1}{2p}} &\lesssim \delta^{-\alpha(p)-\varepsilon}. \end{aligned}$$

For this to hold for all $\delta > 0$, we require

$$\frac{n-1}{4} - \frac{n-1}{2p} \leq \alpha(p),$$

from which we arrive at $p \geq 2\frac{n+1}{n-1}$. ♢

Remark 1.16. The δ exponent on the right-hand side (i.e. $-\alpha(p) - \varepsilon$) is the best possible, except possibly the ε [GS10, p. 1]. ♢

1.2.1 The mixed-norm Wolff inequality

As in [GS10], we are interested in the closely related mixed-norm variant of (1.8),

$$\left\| \sum_j f_j \right\|_p \leq C_\varepsilon \delta^{-\beta(p)-\varepsilon} \left(\sum_j \|f_j\|_p^2 \right)^{1/2}, \tag{1.10}$$

where $\beta(p) = \frac{n-1}{4} - \frac{n+1}{2p}$.

Remark 1.17. The mixed-norm inequality (1.10) for a certain p implies Wolff's inequality (1.8) for the same p, since by Hölder's inequality,

$$\begin{aligned} \left(\sum_j \|f_j\|_p^2 \right)^{1/2} &\leq \left(\left(\sum_j \left(\|f_j\|_p^2 \right)^{p/2} \right)^{2/p} \left(\sum_j 1 \right)^{1-2/p} \right)^{1/2} \\ &\lesssim \delta^{-\frac{n-1}{4}\left(1-\frac{2}{p}\right)} \left(\sum_j \|f_j\|_p^p \right)^{1/p} \end{aligned}$$

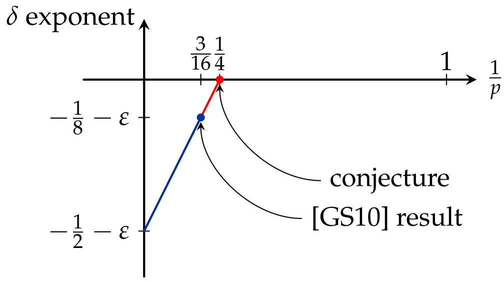
and we have $\beta(p) + \frac{n-1}{4}\left(1 - \frac{2}{p}\right) = \alpha(p)$.

Combining this observation with Remark 1.15 and Remark 1.16, we see that the conjectured range $p \geq 2\frac{n+1}{n-1}$ and exponent $-\beta(p) - \varepsilon$ are the best possible in (1.10). ♢

Currently, the best known result for paraboloids is:

Theorem 1.18 (Garrigós-Seeger, [GS10]). *The inequality* (1.10) *holds for all* $\varepsilon > 0$*, when* $n \geq 2$ *and* $p \geq 2 + \frac{8}{n-1} - \frac{4}{n(n-1)}$.

This is illustrated for the $n = 3$ case in the following diagram:



We note that at the endpoint $p = 2\frac{n+1}{n-1}$ there should be almost no δ-dependence; $A_\delta = C_\varepsilon \delta^{-\varepsilon}$, or perhaps even $A_\delta = O\left(\left(\log \frac{1}{\delta}\right)^N\right)$ for some N. It may even be the case that there is no δ-dependence at all, i.e. $A_\delta = O(1)$.

In Chapter 3, we investigate (1.10) at the endpoint $p = 2\frac{n+1}{n-1}$, specifically when $n = 2, 3$. Our main result is that the inequality cannot hold with $\varepsilon = 0$ in these cases, i.e. A_δ must have some δ-dependence (see §3.2). We then proceed to establish some positive results in a series of case studies (§3.4–§3.6).

Chapter

2 Multilinear Kakeya Question

Our aim in this chapter is to address an important special case of the multilinear Kakeya conjecture — the bush example, which we have already seen is important for the linear problem. We begin these calculations in §2.1, but in order to tackle the case of arbitrary tubes (i.e. (2.3) below), we move to a continuous variant of the question in §2.2. There, we show that the argument used for the linear problem will not suffice, and give a proof for the $n = 2$ case with a view to generalising it. Our main result is then established in §2.3; this deals with a particular part of the arbitrary tubes problem, which is considered to be the main term, when $n = 3$.

We are now concerned with doubly infinite 1-tubes, i.e. 1-neighbourhoods of lines. As before, the direction of the tube T is denoted $e(T) \in \mathbb{S}^{n-1}$.

Let $\mathbf{T}_1, \dots, \mathbf{T}_n$ be families of 1-tubes. We recall the known results stated in §1.1.3; for simplicity, we take any constants $c_T = 1$.

- If the families $\mathbf{T}_j$ are transverse, we have

$$\int_{\mathbb{R}^n} \left(\sum_{T_1 \in \mathbf{T}_1} \chi_{T_1}(x) \cdots \sum_{T_n \in \mathbf{T}_n} \chi_{T_n}(x) \right)^{\frac{1}{n-1}} dx \lesssim (\#\mathbf{T}_1 \dots \#\mathbf{T}_n)^{\frac{1}{n-1}} . \tag{2.1}$$

- If the families $\mathbf{T}_j$ are *quantitatively transverse*, i.e. $e(T_1) \wedge \cdots \wedge e(T_n) \geq \alpha$ for all $T_j \in \mathbf{T}_j$, then

$$\int_{\mathbb{R}^n} \left(\sum_{T_1 \in \mathbf{T}_1} \chi_{T_1}(x) \cdots \sum_{T_n \in \mathbf{T}_n} \chi_{T_n}(x) \right)^{\frac{1}{n-1}} dx \lesssim \alpha^{-\frac{1}{n-1}} (\#\mathbf{T}_1 \dots \#\mathbf{T}_n)^{\frac{1}{n-1}} . \tag{2.2}$$

- For arbitrary families $\mathbf{T}_j$, we have

$$\int_{\mathbb{R}^n} \left(\sum_{T_1 \in \mathbf{T}_1} \chi_{T_1}(x) \cdots \sum_{T_n \in \mathbf{T}_n} \chi_{T_n}(x) e(T_1) \wedge \cdots \wedge e(T_n) \right)^{\frac{1}{n-1}} dx \lesssim (\#\mathbf{T}_1 \ldots \#\mathbf{T}_n)^{\frac{1}{n-1}}. \quad (2.3)$$

2.1 The bush example

In analogy with the linear problem, we expect that requiring every tube to pass through the origin (in fact, to be a 1-neighbourhood of a line through the origin) will make these questions easier to answer and, as in the linear case, be a key point in the development of the theory. Indeed, a central observation of [BCT06] is that in the multilinear transverse case, one may expect the bush example to be the "worst" case (as discussed in [BCT06, Question 1.14]).

The current proof of (2.3) (e.g. in [CV12]) is highly abstract, so the main aim of our approach is to give a hands-on, constructive proof in this case. A secondary aim is to obtain each estimate with a good idea of the constant involved — it has been conjectured (J. Bennett, personal communication) that with suitable normalisation the constant in (2.3) may be 1, as can be obtained in the $n = 2$ case (see §2.2.3).

We will examine this question for each of the inequalities (2.1), (2.2) and (2.3) in the following sub-sections.

A note on $\omega_1 \wedge \cdots \wedge \omega_n$

The following lemma will be useful when dealing with the wedge quantity appearing in (2.2) and (2.3).

Lemma 2.1. *If each $\omega_i \in \mathbb{S}^{n-1}$ makes an angle of at most R with some fixed $u \in \mathbb{S}^{n-1}$, then*

$$\omega_1 \wedge \cdots \wedge \omega_n \leq 2^{n-1} R^{n-1}.$$

Proof. Just as $\omega_1 \wedge \cdots \wedge \omega_n$ is the volume of the parallelepiped with edges ω_i, we can let $\omega_1 \wedge \cdots \wedge \omega_k$ be the k-volume of the parallelotope in $\mathbb{R}^n$ with edges $\omega_1, \ldots, \omega_k$. In this way, $\omega_1 \wedge \cdots \wedge \omega_n$ can be computed inductively; for instance, with $\omega_1, \omega_2, \omega_3 \in \mathbb{S}^2$, we have

$$\omega_1 \wedge \omega_2 \wedge \omega_3 = \omega_1 \wedge \omega_2 \times \left| \omega_3 - P_{\omega_1,\omega_2}(\omega_3) \right|,$$

where P_{ω_1,ω_2} is the orthogonal projection onto the span of ω_1 and ω_2.

- Given $\omega_1, \omega_2 \in \mathbb{S}^{n-1}$, $\omega_1 \wedge \omega_2$ is the area of a parallelogram. If the angle between ω_1 and ω_2 is θ, this area is $\sin\theta \le \theta$. By hypothesis we have $\theta \le 2R$, thus $\omega_1 \wedge \omega_2 \le 2R$.

- Now suppose $\omega_1 \wedge \cdots \wedge \omega_{k-1} \le (2R)^{(k-1)-1}$. We then have

$$\begin{aligned}\omega_1 \wedge \cdots \wedge \omega_k &= \omega_1 \wedge \cdots \wedge \omega_{k-1} \times \left|\omega_k - P_{\omega_1,\ldots,\omega_{k-1}}(\omega_k)\right| \\ &\le (2R)^{(k-1)-1} \times 2R\end{aligned}$$

  since

$$\left|\omega_k - P_{\omega_1,\ldots,\omega_{k-1}}(\omega_k)\right| \le |\omega_k - \omega_1|_{\mathbb{R}^n} \le |\omega_k - \omega_1|_{\mathbb{S}^{n-1}} \le 2R.$$

Thus by induction, we obtain the result. ■

2.1.1 Transverse tubes through the origin

As remarked in [BCT06], if we impose the condition that all the tubes are centred at the origin, the quantities in (2.1) are “trivially comparable” — from the transversality condition, a given $x \in \mathbb{R}^n$ cannot simultaneously lie in tubes $T_1, \ldots, T_n$ if $|x| > 2$ (say), so the left-hand side is clearly

$$\le \operatorname{vol}(|x| \le 2) \left(\prod_{j=1}^n \#\mathbf{T}_j\right)^{\frac{1}{n-1}} \lesssim \text{RHS}.$$

2.1.2 Quantitatively transverse tubes through the origin

Rewriting the integral using polar coordinates, with dyadic ranges of radii,

$$\text{LHS (2.2)} \lesssim \sum_{k=0}^{\infty} \underbrace{\int_{r\sim 2^k} r^{n-1}\, dr}_{\sim 2^{kn}} \int_{u\in\mathbb{S}^{n-1}} \left(\sum_{T_1\in\mathbf{T}_1} \chi_{T_1}(2^k u) \cdots \sum_{T_n\in\mathbf{T}_n} \chi_{T_n}(2^k u)\right)^{\frac{1}{n-1}} d\sigma(u).$$

Now to have $2^k u \in T_1, \ldots, T_n$, we must have $e(T_j) \in \operatorname{cap}_{2^{-k}}(u)$, so it follows from Lemma 2.1 that $e(T_1) \wedge \cdots \wedge e(T_n) \lesssim (2^{-k})^{n-1}$. From the transversality condition, we see that there can be no such tubes if $2^{-k(n-1)} \le c\alpha$, i.e. if $k \ge \log_2(c'\alpha^{-\frac{1}{n-1}}) =: \lg A$. So the sum in k is finite.

Furthermore, we can break up the u integral using a finitely overlapping covering of

$\mathbb{S}^{n-1}$ by caps of radius 2^{-k} indexed by u_ℓ, obtaining

$$\begin{aligned}\text{LHS (2.2)} &\lesssim \sum_{k=0}^{\lg A} 2^{kn} \sum_{\ell} \int_{u \in \mathrm{cap}_{2^{-k}}(u_\ell)} \left(\sum_{T_1 \in \mathbf{T}_1} \chi_{T_1}(2^k u) \cdots \sum_{T_n \in \mathbf{T}_n} \chi_{T_n}(2^k u) \right)^{\frac{1}{n-1}} d\sigma(u) \\ &\lesssim \sum_{k=0}^{\lg A} 2^{kn} \sum_{\ell} \sigma\left(\mathrm{cap}_{2^{-k}}(u_\ell)\right) \max_{u \in \mathrm{cap}_{2^{-k}}(u_\ell)} \left(\prod_{j=1}^{n} \sum_{T_j \in \mathbf{T}_j} \chi_{T_j}(2^k u) \right)^{\frac{1}{n-1}} \\ &\lesssim \sum_{k=0}^{\lg A} 2^{k} \sum_{\ell} \left(a_1(\ell) \cdots a_n(\ell)\right)^{\frac{1}{n-1}} \qquad (2.4)\end{aligned}$$

where we have defined $a_j(\ell) = \max_{u \in \mathrm{cap}_{2^{-k}}(u_\ell)} \sum_{T_j \in \mathbf{T}_j} \chi_{T_j}(2^k u)$.

We now need the following two facts:

Lemma 2.2. *For any $a_j(\ell) \geq 0$,*

$$\sum_{\ell} \left(a_1(\ell) \cdots a_n(\ell)\right)^{\frac{1}{n-1}} \leq \left(\sum_{\ell} a_1(\ell) \cdots \sum_{\ell} a_n(\ell) \right)^{\frac{1}{n-1}}. \qquad (2.5)$$

Proof. Applying Hölder's inequality,

$$\sum_{\ell} a_1(\ell)^{\frac{1}{n-1}} \cdots a_n(\ell)^{\frac{1}{n-1}} \leq \left(\sum_{\ell} a_1(\ell)^{\frac{n}{n-1}} \right)^{\frac{1}{n}} \cdots \left(\sum_{\ell} a_n(\ell)^{\frac{n}{n-1}} \right)^{\frac{1}{n}}.$$

Now note that

$$\left(\sum_{\ell} a_j(\ell)^{\frac{n}{n-1}} \right)^{\frac{1}{n}} = \left(\sum_{\ell} a_j(\ell)^{\frac{n}{n-1}} \right)^{\frac{n-1}{n} \frac{1}{n-1}} = \|a_j\|_{\ell^{\frac{n}{n-1}}}^{\frac{1}{n-1}} \leq \|a_j\|_{\ell^1}^{\frac{1}{n-1}},$$

since $\|\cdot\|_{\ell^p} \leq \|\cdot\|_{\ell^1}$ for $p \geq 1$ and we have $\frac{n}{n-1} > 1$. ∎

Remark 2.3. Equality can only be attained in (2.5) when both the inequalities applied in the proof are in fact equalities. Thus, due to the application of Hölder's inequality, this requires $a_1 = \cdots = a_n$, and from the ℓ^1-$\ell^{n/(n-1)}$ embedding we require $a_j(\ell) \neq 0$ for just one ℓ.

Given how restrictive these conditions are, we see that (2.5) is not very efficient. ◊

Lemma 2.4. $\sum_{\ell} a_j(\ell) \lesssim \#\mathbf{T}_j$

Proof. We get a contribution to $a_j(\ell)$ from a particular tube T_j if there is a $2^k u \in 2^k \mathrm{cap}_{2^{-k}}(u_\ell)$ which also lies in T_j (where $2^k \mathrm{cap}_{2^{-k}}(u_\ell)$ is a subset of $2^k \mathbb{S}^{n-1}$ with aperture 2^{-k}, i.e. radius 1). Thus

$$a_j(\ell) \leq \text{number of } T_j \in \mathbf{T}_j \text{ which intersect } 2^k \mathrm{cap}_{2^{-k}}(u_\ell).$$

Note that this cap can also be viewed as a cap of radius 1 on $2^k \mathrm{S}^{n-1}$. Since T_j has radius 1 and (crucially) passes through the origin, it can overlap $O(1)$ such caps, hence the sum will overcount each tube at most $O(1)$ times. ■

Using these facts, we get

$$\text{LHS (2.2)} \lesssim \sum_{k=0}^{\lg A} 2^k \left(\#\mathbf{T}_1 \cdots \#\mathbf{T}_n\right)^{\frac{1}{n-1}} \lesssim \alpha^{-\frac{1}{n-1}} \left(\#\mathbf{T}_1 \cdots \#\mathbf{T}_n\right)^{\frac{1}{n-1}}.$$

Thus,

Theorem 2.5. *If all $T \in \mathbf{T}_j$ are neighbourhoods of lines through the origin, then* (2.2) *holds.*

2.1.3 Arbitrary tubes through the origin

This case is not as easy to deal with as those already considered, but we begin in a similar way.

We define the subset W_j of $\mathbf{T}_1 \times \cdots \times \mathbf{T}_n$ as

$$W_j = \left\{ (T_1, \ldots, T_n) \,:\, e(T_1) \wedge \cdots \wedge e(T_n) \sim 2^{-j(n-1)} \right\}.$$

For concreteness, we suppose $e(T_1) \wedge \cdots \wedge e(T_n) \sim 2^{-j(n-1)}$ means

$$\begin{aligned} & e(T_1) \wedge \cdots \wedge e(T_n) \leq 2^{2(n-1)} 2^{-j(n-1)} \\ \text{and} \quad & e(T_1) \wedge \cdots \wedge e(T_n) \geq 2^{(n-1)} 2^{-j(n-1)}. \end{aligned} \tag{2.6}$$

Then

$$\text{LHS (2.3)} = \int_{\mathbb{R}^n} \left(\sum_{j=0}^{\infty} \sum_{(T_1,\ldots,T_n) \in W_j} \chi_{T_1}(x) \cdots \chi_{T_n}(x) e(T_1) \wedge \cdots \wedge e(T_n) \right)^{\frac{1}{n-1}} dx,$$

and repeating the first steps in the previous section, this is bounded by

$$\sum_{k=0}^{\infty} 2^{kn} \sum_{\ell} |\mathrm{cap}_{2^{-k}}(u_\ell)| \max_{u \in \mathrm{cap}_{2^{-k}}(u_\ell)} \left(\sum_{j=0}^{\infty} \sum_{(T_1,\ldots,T_n) \in W_j} \chi_{T_1}(2^k u) \cdots \chi_{T_n}(2^k u) 2^{-j(n-1)} \right)^{\frac{1}{n-1}}.$$

Note that $\chi_{T_1}(2^k u)\cdots\chi_{T_n}(2^k u)$ can only be nonzero if the directions of the T_i all lie within 2^{-k} of u on $\mathbb{S}^{n-1}$, so by Lemma 2.1 we know $e(T_1)\wedge\cdots\wedge e(T_n)\le 2^{n-1}2^{-k(n-1)}$. Thus, due to our choice of constants in (2.6), the only nonzero contributions come from $j\ge k$. Now writing

$$W_j(\ell)=\left\{(T_1,\ldots,T_n)\in W_j\,:\,e(T_i)\in \mathrm{cap}_{2\times 2^{-k}}(u_\ell)\right\}$$

we have

$$\text{LHS (2.3)}\lesssim\sum_{k=0}^{\infty}2^k\sum_{\ell}\left(\sum_{j=k}^{\infty}\#W_j(\ell)2^{-j(n-1)}\right)^{\frac{1}{n-1}}.$$

- Note that because for each k the u_ℓ are chosen to index a finitely overlapping covering of $\mathbb{S}^{n-1}$ by 2^{-k}-caps, we have

  $$\sum_{\ell}\#W_j(\ell)\lesssim\#W_j\quad\text{for each } j \text{ and } k,$$

  with the implied constant independent of j and k.

- But note that we do not necessarily get all of W_j from $\cup_\ell W_j(\ell)$.

  This is because Lemma 2.1 only goes one way — so there may be $(T_1,\ldots,T_n)\in W_j$ with $e(T_1)\wedge\cdots\wedge e(T_n)\sim 2^{-j(n-1)}$ but not because all the $e(T_i)$ lie in the same 2^{-j}-cap, or indeed even the same 2×2^{-k}-cap.

The $n=2$ case We can move the sums around to get

$$\sum_{j=0}^{\infty}2^{-j}\sum_{k\le j}2^k\sum_{\ell}\#W_j(\ell)\lesssim\sum_{j=0}^{\infty}2^{-j}2^j\#W_j\lesssim\#\mathbf{T}_1\ldots\#\mathbf{T}_n.$$

For $n>2$ We need to show:

Conjecture 2.6.

$$\sum_{k=0}^{\infty}2^k\sum_{\ell}\left(\sum_{j=k}^{\infty}\#W_j(\ell)2^{-j(n-1)}\right)^{\frac{1}{n-1}}\lesssim\left(\sum_{j=0}^{\infty}\#W_j\right)^{\frac{1}{n-1}}.$$

In order to make progess, we move on to consider another version of the problem.

2.2 Continuous Multilinear Kakeya Problem

As noted in a footnote of [CV12, p2], the continuous analogue of (2.3) is

$$\int_{\mathbb{R}^n} \left(\int \cdots \int_{(\mathbb{S}^{n-1})^n} \prod_{j=1}^n g_j(\omega_j, \pi_{\omega_j} x) \omega_1 \wedge \cdots \wedge \omega_n \, d\sigma(\omega_1) \cdots d\sigma(\omega_n) \right)^{\frac{1}{n-1}} dx \\ \leq C_n \prod_{j=1}^n \left(\int_{\mathbb{S}^{n-1} \times \mathbb{R}^{n-1}} g_j(\omega, \pi_\omega x) \right)^{\frac{1}{n-1}}, \quad (2.7)$$

where π_ω is the projection onto the plane perpendicular to ω passing through the origin, and we suppose the g_j are nonnegative.

This is in fact equivalent to the following more general form of (2.3)

$$\int_{\mathbb{R}^n} \left(\sum_{T_1 \in \mathbf{T}_1} a_{T_1} \chi_{T_1}(x) \cdots \sum_{T_n \in \mathbf{T}_n} a_{T_n} \chi_{T_n}(x) e(T_1) \wedge \cdots \wedge e(T_n) \right)^{\frac{1}{n-1}} dx \\ \lesssim \left(\sum_{T_1 \in \mathbf{T}_1} a_{T_1} \cdots \sum_{T_n \in \mathbf{T}_n} a_{T_n} \right)^{\frac{1}{n-1}}, \quad (2.8)$$

with the $a_T \geq 0$, and also to

$$\int_{\mathbb{R}^n} \left(\int \cdots \int_{(\mathbb{S}^{n-1})^n} \prod_{j=1}^n \chi_{T_{\omega_j}}(x) f_j(\omega_j) \omega_1 \wedge \cdots \wedge \omega_n \, d\sigma(\omega_1) \cdots d\sigma(\omega_n) \right)^{\frac{1}{n-1}} dx \\ \leq K_n \prod_{j=1}^n \left(\int_{\mathbb{S}^{n-1}} f_j(\omega) \right)^{\frac{1}{n-1}}, \quad (2.9)$$

where the f_j are nonnegative.

Proof of equivalence.

(2.7) $\Rightarrow$ (2.9) Putting $g_j(\omega_j, \pi_{\omega_j} x) = \chi_{T_{\omega_j}}(x) f_j(\omega_j)$, we have

$$\text{LHS (2.9)} = \text{LHS (2.7)} \leq C_n \prod_{j=1}^n \left(\int_{\mathbb{S}^{n-1} \times \mathbb{R}^{n-1}} \chi_{T_{\omega_j}}(x) f_j(\omega_j) \right)^{\frac{1}{n-1}}.$$

Thus we obtain the right-hand side of (2.9), with

$$K_n = \Big(\text{cross-sectional area of tubes}\Big)^{\frac{n}{n-1}} C_n.$$

(2.9) $\Rightarrow$ (2.8) Let

$$f_j(\omega_j) = \sum_{T_j \in \mathbf{T}_j} a_{T_j} \delta_{e(T_j)}(\omega_j),$$

where $\delta_{e(T_j)}$ is the Dirac delta function at $e(T_j)$ on $\mathbb{S}^{n-1}$.

Putting these into (2.9) gives (2.8).

(2.8) $\Rightarrow$ (2.7) It suffices to show (2.7) for step-functions $g_j(\omega_j, \pi_{\omega_j} x)$, and then take a limit as the maximum length of each step (which we denote t) tends to 0.

We approximate the integral over $\mathbb{S}^{n-1}$ by a sum over representative ω from each step. Then for each fixed ω, observe that $g(\omega, \pi_\omega x)$ is a function of $x \in \mathbb{R}^{n-1}$ and, since it is a step-function, it can be written as

$$\sum_\alpha g^\alpha(\omega) \chi_{T^\alpha_{\omega,t}}(x)$$

where the $T^\alpha_{\omega,t}$ are t-tubes in the direction ω (i.e. tubes of width t), with associated constants $g^\alpha(\omega)$. Thus the left-hand side of (2.7) is approximated by

$$\int_{\mathbb{R}^n} \left(\sum_{\omega_1} \sum_{\alpha_1} g_1^{\alpha_1}(\omega_1) \chi_{T^{\alpha_1}_{\omega_1,t}}(x) \cdots \sum_{\omega_n} \sum_{\alpha_n} g_n^{\alpha_n}(\omega_n) \chi_{T^{\alpha_n}_{\omega_n,t}}(x)\ \omega_1 \wedge \cdots \wedge \omega_n \right)^{\frac{1}{n-1}} dx,$$

and after dilating ($x \mapsto \frac{x}{t}$), we obtain the left-hand side of (2.8) with $\mathbf{T}_j = \left\{ T^{\alpha_j}_{\omega_j,1} \right\}$, multiplied by t^n. So, by (2.8), this is bounded by

$$\prod_{j=1}^n \left(\sum_{\omega_j} t^{n-1} \sum_{\alpha_j} g_j^{\alpha_j}(\omega_j) \right)^{\frac{1}{n-1}} = \prod_{j=1}^n \left(\sum_{\omega_j} \int_{\mathbb{R}^{n-1}} g_j(\omega_j, \pi_{\omega_j} x)\ dx \right)^{\frac{1}{n-1}}$$

and this, in the limit, is the right-hand side of (2.7). ■

Now (2.7) is known to be true by [CV12], so the special case where all the tubes are centred at the origin is also true. However, we seek to prove this special case more directly.

2.2.1 Some reductions

Characteristic functions

We can simplify the problem slightly by supposing the functions g_i are given by characteristic functions of sets, i.e. $g_j(\omega_j, \pi_{\omega_j} x) = \chi_{E_j}(\omega_j) \chi_{T_{\omega_j}}(x)$, where $E_j \subseteq \mathbb{S}^{n-1}$.

Put $E = E_1 \times \cdots \times E_n$, and for $x \in \mathbb{R}^n$ let

$$T^{-1}(x) = \{\omega \in \mathbb{S}^{n-1} \,:\, T_\omega \ni x\},$$

with the T_ω being tubes through the origin (thus $T^{-1}(x) \subset \mathrm{cap}_{1/|x|}\,(x/|x|)$).

In this special case, inequality (2.7) becomes:

$$\int_{\mathbb{R}^n} \left(\int \ldots \int_{E \cap (T^{-1}(x))^n} \omega_1 \wedge \cdots \wedge \omega_n \; d\sigma(\omega_1) \cdots d\sigma(\omega_n) \right)^{\frac{1}{n-1}} dx \leq C_n |E|^{\frac{1}{n-1}}. \qquad (2.10)$$

Away from the origin

Note that in (2.9), the contribution to the x integral from $\{x \in \mathbb{R}^n \,:\, |x| \leq R\}$ is bounded by

$$\int_{|x| \leq R} \left(\int \cdots \int_{(\mathbb{S}^{n-1})^n} \prod_{j=1}^{n} f_j(\omega_j) \; d\sigma(\omega_1) \cdots d\sigma(\omega_n) \right)^{\frac{1}{n-1}} dx$$

$$\lesssim R^n \prod_{j=1}^{n} \left(\int_{\mathbb{S}^{n-1}} f_j(\omega) \right)^{\frac{1}{n-1}}.$$

since $\chi_{T_{\omega_1}}(x) \cdots \chi_{T_{\omega_1}}(x) \omega_1 \wedge \cdots \wedge \omega_n \leq 1$.

Flattening out

It remains to consider the contribution to (2.9) away from the origin, i.e. with $|x| > R$.

We first note that it suffices to consider a small set of tube directions in $\mathbb{S}^{n-1}$. Suppose $\alpha_1, \ldots, \alpha_J \in \mathbb{S}^{n-1}$ index a J_0-overlapping covering of $\mathbb{S}^{n-1}$ by caps of the form $\mathrm{cap}_{1/2}(\alpha_i)$. Then each $f_j \leq \sum_\alpha f_j \chi_{\mathrm{cap}_{1/2}(\alpha)}$, so we can bound the contribution to the left-hand side of (2.9) by

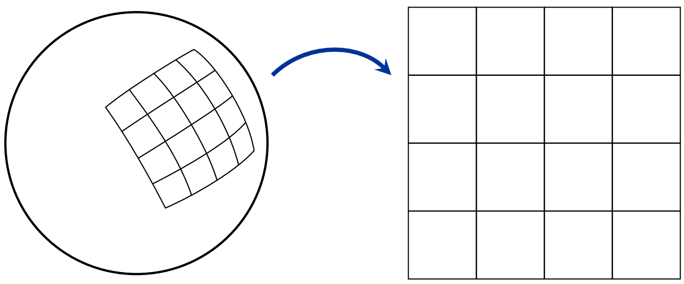
$$\sum_\alpha \int_{|x| > R} \left(\int \cdots \int_{(\mathrm{cap}_{1/2}(\alpha))^n} \prod_{j=1}^{n} \chi_{T_{\omega_j}}(x) f_j(\omega_j) \omega_1 \wedge \cdots \wedge \omega_n \; d\sigma(\omega_1) \cdots d\sigma(\omega_n) \right)^{\frac{1}{n-1}} dx.$$

Now if we have the desired result for each term in the α sum, this gives the bound

$$\begin{aligned}
&\sum_\alpha K_n \left(\int_{\mathrm{cap}_{1/2}(\alpha)} f_1(\omega)\, d\sigma(\omega) \cdots \int_{\mathrm{cap}_{1/2}(\alpha)} f_n(\omega)\, d\sigma(\omega) \right)^{\frac{1}{n-1}} \\
&\leq K_n J^{\frac{n-2}{n-1}} \left(\sum_\alpha \int_{\mathrm{cap}_{1/2}(\alpha)} f_1(\omega)\, d\sigma(\omega) \cdots \int_{\mathrm{cap}_{1/2}(\alpha)} f_n(\omega)\, d\sigma(\omega) \right)^{\frac{1}{n-1}} \\
&\leq K_n J^{\frac{n-2}{n-1}} J_0^{\frac{1}{n-1}} \left(\int_{\mathbb{S}^{n-1}} f_1(\omega)\, d\sigma(\omega) \cdots \int_{\mathbb{S}^{n-1}} f_n(\omega)\, d\sigma(\omega) \right)^{\frac{1}{n-1}},
\end{aligned}$$

from an application of Hölder's inequality.

Thus we restrict attention to a subset $U \subset \mathbb{S}^{n-1}$ which can be flattened out to give $[0,1]^{n-1}$, via (for instance) a suitable stereographic projection $P : U \to [0,1]^{n-1}$.



Upon changing variables, we see that (2.9) would follow from

$$\int_{|x|>R} \left(\int_{([0,1]^{n-1})^n} \cdots \int \prod_{j=1}^{n} \chi_{T_{\omega_j}}(x) f_j(\omega_j) \omega_1 \wedge \cdots \wedge \omega_n \, d\omega_1 \cdots d\omega_n \right)^{\frac{1}{n-1}} dx$$
$$\leq C_P K_n \prod_{j=1}^{n} \left(\int_{[0,1]^{n-1}} f_j(\omega)\, d\omega \right)^{\frac{1}{n-1}}, \quad (2.11)$$

where C_P is a constant due to the change of variables, depending on the choice of projection P. Note that we now have $\omega_1, \ldots \omega_n \in [0,1]^{n-1}$ (i.e. no longer on $\mathbb{S}^{n-1}$), so $\omega_1 \wedge \cdots \wedge \omega_n$ should be interpreted as the volume of the parallelepiped with edges $\langle \omega_j, 1 \rangle \in \mathbb{R}^n$, and T_{ω_j} as the tube in the direction $\langle \omega_j, 1 \rangle$.

Constant at scale M

With a view to a proof by induction on scales, we observe that it suffices to consider a special class of functions.

Definition 2.7 (Constant at scale M)**.** We shall say a function is constant at scale M if there is a partition of its domain into sets of diameter $\sim 2^{-M}$ with f constant on each of these sets. $\Diamond$

Lemma 2.8. *It suffices to prove* (2.9) *for $f_1, \ldots, f_n$ constant at scale M, for M arbitrarily large, and constant independent of M.*

Proof. Let $F_R(f_1, \ldots, f_n)$ denote the left-hand side of (2.9), but with the x-integral over $|x| \leq R$. Our aim is to show that, for arbitrary $f_j : \mathbb{S}^{n-1} \to \mathbb{R}_{\geq 0}$, we have

$$F_R(f_1, \ldots, f_n) \leq C_R \left(\|f_1\|_1 \cdots \|f_n\|_1\right)^{\frac{1}{n-1}}$$

with the constant C_R bounded as $R \to \infty$.

Fix $\varepsilon > 0$. Given R, choose a sufficiently large integer $M = M(R)$, and functions $f_j^{(M)} : \mathbb{S}^{n-1} \to \mathbb{R}_{\geq 0}$ such that

(1) each $f_j^{(M)}$ is constant at scale M, and

(2) $\|f_j^{(M)} - f_j\|_1 \leq \left(\frac{\varepsilon}{C_n R^n}\right)^{n-1} \|f_j\|_1$, where $C_n = R^{-n} \int_{|x| \leq R} dx$.

Note that (2) gives

$$\|f_j^{(M)}\|_1 \leq \|f_j^{(M)} - f_j\|_1 + \|f_j\|_1 \leq \left(1 + \left(\frac{\varepsilon}{C_n R^n}\right)^{n-1}\right) \|f_j\|_1. \tag{2'}$$

Now $F_R(f_1, \ldots, f_n)$ can be written as

$$\begin{aligned}
&F_R\left((f_1 - f_1^{(M)}) + f_1^{(M)}, \ldots, (f_n - f_n^{(M)}) + f_n^{(M)}\right) \\
&= \int_{|x| \leq R} \left(\int \cdots \int_{(\mathbb{S}^{n-1})^n} \prod_{j=1}^{n} \chi_{T_{\omega_j}}(x) \left((f_j - f_j^{(M)}) + f_j^{(M)}\right) \omega_1 \wedge \cdots \wedge \omega_n \, d\sigma(\omega_1) \cdots d\sigma(\omega_n) \right)^{\frac{1}{n-1}} dx \\
&\leq F_R\left(f_1 - f_1^{(M)}, \ldots, f_n - f_n^{(M)}\right) + \cdots + F_R\left(f_1^{(M)}, \ldots, f_n^{(M)}\right),
\end{aligned} \tag{2.12}$$

after multiplying out the n brackets and using $(\sum x_i)^{\frac{1}{n-1}} \leq \sum x_i^{\frac{1}{n-1}}$. This leaves 2^n terms to consider, according to whether each factor is $(f_j - f_j^{(M)})$ or $f_j^{(M)}$.

- For $F_R\left(f_1^{(M)},\ldots,f_n^{(M)}\right)$, each function is constant at scale M, so by assumption (2.9) holds with $K_n = C_{M,n}$. Thus

$$F_R\left(f_1^{(M)},\ldots,f_n^{(M)}\right) \le C_{M,n}\left(\|f_1^{(M)}\|_1\cdots\|f_n^{(M)}\|_1\right)^{\frac{1}{n-1}}$$
$$\le C_{M,n}\left(1+\left(\frac{\varepsilon}{C_nR^n}\right)^{n-1}\right)^n\left(\|f_1\|_1\cdots\|f_n\|_1\right)^{\frac{1}{n-1}}$$

  from (2′).

- There are n terms where only one of the factors is of the form $(f_j - f_j^{(M)})$; for instance

$$F_R\left(f_1 - f_1^{(M)}, f_2^{(M)},\ldots,f_n^{(M)}\right)$$
$$\le \int_{|x|\le R}\left(\int\cdots\int_{(\mathbb{S}^{n-1})^n}\left(\left(f_1 - f_1^{(M)}\right)\right)f_2^{(M)}\cdots f_n^{(M)}\,d\sigma(\omega_1)\cdots d\sigma(\omega_n)\right)^{\frac{1}{n-1}}dx$$

  since $\chi_{T_{\omega_1}}\cdots\chi_{T_{\omega_n}}\omega_1\wedge\cdots\wedge\omega_n \le 1$. Now this is bounded by

$$C_nR^n\left(\|f_1 - f_1^{(M)}\|_1\|f_2^{(M)}\|_1\cdots\|f_n^{(M)}\|_1\right)^{\frac{1}{n-1}}$$
$$\le \varepsilon\left(1+\left(\frac{\varepsilon}{C_nR^n}\right)^{n-1}\right)^{n-1}\left(\|f_1\|_1\cdots\|f_n\|_1\right)^{\frac{1}{n-1}},$$

  by (2) and (2′).

- All the remaining terms have $k>1$ factors of the form $(f_j - f_j^{(M)})$, so repeating the argument above will yield

$$C_nR^n\left(\frac{\varepsilon}{C_nR^n}\right)^k\left(1+\left(\frac{\varepsilon}{C_nR^n}\right)^{n-1}\right)^{n-k}\left(\|f_1\|_1\cdots\|f_n\|_1\right)^{\frac{1}{n-1}}$$

  which has constant $O(R^{-n(k-1)})$, which is certainly $O(R^{-n})$.

Putting all of these estimates into (2.12),

$$F_R(f_1,\ldots,f_n)$$
$$\le\left(C_{M,n}\left(1+\left(\frac{\varepsilon}{C_nR^n}\right)^{n-1}\right)^n + n\varepsilon\left(1+\left(\frac{\varepsilon}{C_nR^n}\right)^{n-1}\right)^{n-1} + O(R^{-n})\right)\left(\|f_1\|_1\cdots\|f_n\|_1\right)^{\frac{1}{n-1}}.$$

Since we assume $C_{M,n} = C'_n$ (i.e. there is no dependence on M, and hence R), we can let $R \to \infty$ in the above estimate and obtain

$$F_R(f_1, \dots, f_n) \leq \left(C'_n + n\varepsilon\right) \left(\|f_1\|_1 \cdots \|f_n\|_1\right)^{\frac{1}{n-1}}.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$F_R(f_1, \dots, f_n) \leq C'_n \left(\|f_1\|_1 \cdots \|f_n\|_1\right)^{\frac{1}{n-1}}. \qquad \blacksquare$$

Remark 2.9. This shows that the same constant we get for f_i "constant at scale M" will work for arbitrary f_i. $\Diamond$

2.2.2 Quantitatively transverse tubes

As with the discrete case, we can consider a simpler inequality than (2.10), by only allowing tubes which are somewhat transverse (this is the analogue of (2.2)).

Proposition 2.10. *If we assume that, for some fixed (small) $\alpha > 0$,*

$$\omega_i \in E_i \Rightarrow \omega_1 \wedge \cdots \wedge \omega_n \geq \alpha$$

then

$$\int_{\mathbb{R}^n} \left(\int \cdots \int_{E \cap (T^{-1}(x))^n} d\sigma(\omega_1) \cdots d\sigma(\omega_n) \right)^{\frac{1}{n-1}} dx \leq C_n \alpha^{-\frac{1}{n-1}} |E|^{\frac{1}{n-1}}. \tag{2.13}$$

Proof. Note that the usual decomposition can be carried out, e.g. as in §2.1.2, to show that

$$\text{LHS (2.13)} \lesssim \sum_{k=1}^{\infty} 2^{kn} \sum_{\ell} \int_{u \in \text{cap}_{2^{-k}}(u_\ell)} \left(|E_1 \cap T^{-1}(2^k u)| \cdots |E_n \cap T^{-1}(2^k u)| \right)^{\frac{1}{n-1}} d\sigma(u).$$

Repeating the argument used in (2.4) for the discrete case, this becomes

$$\text{LHS (2.13)} \lesssim \sum_{k=1}^{\lg \alpha^{-\frac{1}{n-1}}} 2^k \sum_{\ell} \left(\prod_{i=1}^{n} \max_{u \in \text{cap}_{2^{-k}}(u_\ell)} |E_i \cap T^{-1}(2^k u)| \right)^{\frac{1}{n-1}},$$

which, by (2.5), is bounded by

$$\sum_{k=1}^{\lg \alpha^{-\frac{1}{n-1}}} 2^k |E|^{\frac{1}{n-1}} \lesssim \alpha^{-\frac{1}{n-1}} |E|^{\frac{1}{n-1}}. \qquad \blacksquare$$

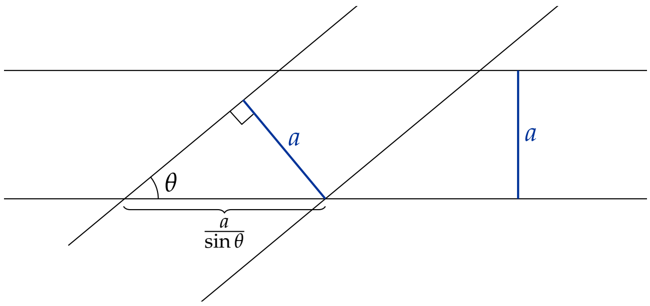
2.2.3 Trivial $n = 2$ case

When $n = 2$ we have $\frac{1}{n-1} = 1$. This means that in (2.9), for instance, we can interchange integrals to obtain

$$\int_{\mathbb{R}^2}\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} f_1(\omega_1)f_2(\omega_2)\,\chi_{T_{\omega_1}}(x)\chi_{T_{\omega_2}}(x)\;\omega_1\wedge\omega_2\;d\sigma(\omega_1)d\sigma(\omega_2)\,dx$$
$$= \int_{\mathbb{S}^1}\int_{\mathbb{S}^1} f_1(\omega_1)f_2(\omega_2)\;\underbrace{\omega_1\wedge\omega_2}_{A}\;\underbrace{\int_{\mathbb{R}^2}\chi_{T_{\omega_1}}(x)\chi_{T_{\omega_2}}(x)\,dx}_{B}\;d\sigma(\omega_1)d\sigma(\omega_2)$$

Now consider a fixed $\omega_1, \omega_2 \in \mathbb{S}^1$ with angle θ between them. Observe that

- $A = |\sin\theta|$, and
- $B = |T_{\omega_1} \cap T_{\omega_2}|$, and this intersection is a parallelogram.



Some basic trigonometry allows us to compute $B = \frac{a}{\sin\theta} \times a$.
So choosing $a = 1$, we have $B = \dfrac{1}{|\sin\theta|}$.

Thus A and B cancel and we are left with

$$\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} f_1(\omega_1)f_2(\omega_2)\,d\sigma(\omega_1)d\sigma(\omega_2) = 1 \times \|f_1\|_1\|f_2\|_1.$$

2.2.4 Using the linear argument

The following argument tries to replicate as far as possible the argument used in the proof of Observation 1.6.

Using polar coordinates on the left-hand side of (2.10), with the radii broken into dyadic ranges and the spherical integration expressed as a sum of integrals over finitely overlapping caps,

$$\text{LHS (2.10)} \lesssim \sum_{k=1}^{\infty} 2^{kn} \sum_{\ell} \int_{u \in \text{cap}_{2^{-k}}(u_\ell)} \left(\int_{E \cap (\text{cap}_{2^{-k}}(u))^n} \cdots \int \omega_1 \wedge \cdots \wedge \omega_n \, d\sigma(\omega_1) \cdots d\sigma(\omega_n) \right)^{\frac{1}{n-1}} d\sigma(u).$$

For $u \in \text{cap}_{2^{-k}}(u_\ell)$,

$$E \cap (\text{cap}_{2^{-k}}(u))^n \subseteq E \cap (\text{cap}_{2 \times 2^{-k}}(u_\ell))^n$$

therefore

$$\text{LHS (2.10)} \lesssim \sum_{k=1}^{\infty} 2^{kn} \sum_{\ell} |\text{cap}_{2^{-k}}(u_\ell)| \left(\int_{E \cap (\text{cap}_{2 \times 2^{-k}}(u_\ell))^n} \cdots \int \omega_1 \wedge \cdots \wedge \omega_n \, d\sigma(\omega_1) \cdots d\sigma(\omega_n) \right)^{\frac{1}{n-1}},$$

which simplifies to give

$$\text{LHS (2.10)} \lesssim \sum_{k=1}^{\infty} 2^{k} \sum_{\ell} \left(\int_{E \cap (\text{cap}_{2 \times 2^{-k}}(u_\ell))^n} \cdots \int \omega_1 \wedge \cdots \wedge \omega_n \, d\sigma(\omega_1) \cdots d\sigma(\omega_n) \right)^{\frac{1}{n-1}}. \tag{2.14}$$

The following result, which arose from a discussion with Jim Wright, shows that the decomposition in (2.14) needs to go further.

Proposition 2.11. *Using Lemma 2.1 on* (2.14) *directly shows*

$$\mathit{LHS}\ (2.10) \lesssim \sum_{k=1}^{\infty} \sum_{\ell} \left(|E_1 \cap \text{cap}_{2 \times 2^{-k}}(u_\ell)| \times \cdots \times |E_n \cap \text{cap}_{2 \times 2^{-k}}(u_\ell)| \right)^{\frac{1}{n-1}} =: L$$

and we can establish

$$L \lesssim |E|^{\frac{1}{n-1}} \log(1/|E|) \quad \textit{and} \quad L \gtrsim |E|^{\frac{1}{n-1}} \log(1/|E|) \textit{ for some } E,$$

i.e. that using Lemma 2.1 on (2.14) *leads to a logarithmic loss.*

Proof. We break the sum L into two parts,

$$L = \sum_{k \text{ s.t. } 2^k \geq \frac{1}{|E|^{1/n(n-1)}}} \sum_{\ell} (\cdots)^{\frac{1}{n-1}} + \sum_{k \text{ s.t. } 2^k < \frac{1}{|E|^{1/n(n-1)}}} \sum_{\ell} (\cdots)^{\frac{1}{n-1}} =: L_1 + L_2.$$

For L_1, we use the fact that $|E_j \cap \mathrm{cap}_{2\times 2^{-k}}(u_\ell)| \lesssim 2^{-k(n-1)}$, which gives

$$\begin{aligned} |E_j \cap \mathrm{cap}_{2\times 2^{-k}}(u_\ell)|^{\frac{1}{n-1}} &= |E_j \cap \mathrm{cap}_{2\times 2^{-k}}(u_\ell)|^{\frac{1}{n(n-1)}} |E_j \cap \mathrm{cap}_{2\times 2^{-k}}(u_\ell)|^{\frac{1}{n}} \\ &\lesssim 2^{-k/n} |E_j \cap \mathrm{cap}_{2\times 2^{-k}}(u_\ell)|^{\frac{1}{n}}. \end{aligned}$$

We then have

$$\begin{aligned} L_1 &\leq \sum_{k \text{ s.t. } 2^k \geq \frac{1}{|E|^{1/n(n-1)}}} 2^{-k} \sum_\ell \left(|E_1 \cap \mathrm{cap}_{2\times 2^{-k}}(u_\ell)| \times \cdots \times |E_n \cap \mathrm{cap}_{2\times 2^{-k}}(u_\ell)| \right)^{\frac{1}{n}} \\ &\leq \sum_{k \text{ s.t. } 2^k \geq \frac{1}{|E|^{1/n(n-1)}}} 2^{-k} \left(\sum_\ell |E_1 \cap \mathrm{cap}_{2\times 2^{-k}}(u_\ell)| \right)^{\frac{1}{n}} \times \cdots \times \left(\sum_\ell |E_n \cap \mathrm{cap}_{2\times 2^{-k}}(u_\ell)| \right)^{\frac{1}{n}} \end{aligned}$$

by Hölder's inequality. Now since ℓ is indexing a finitely overlapping covering of $\mathbb{S}^{n-1}$, this gives

$$\begin{aligned} L_1 &\lesssim \sum_{k \text{ s.t. } 2^k \geq \frac{1}{|E|^{1/n(n-1)}}} 2^{-k} \left(|E_1| \cdots |E_n| \right)^{\frac{1}{n}} \\ &\lesssim |E|^{\frac{1}{n(n-1)}} |E|^{\frac{1}{n}} = |E|^{\frac{1}{n-1}}. \end{aligned}$$

For L_2 we make use of (2.5) to get

$$\begin{aligned} L_2 &\leq \sum_{k \text{ s.t. } 2^k < \frac{1}{|E|^{1/n(n-1)}}} \left(\sum_\ell |E_1 \cap \mathrm{cap}_{2\times 2^{-k}}(u_\ell)| \times \cdots \times \sum_\ell |E_n \cap \mathrm{cap}_{2\times 2^{-k}}(u_\ell)| \right)^{\frac{1}{n-1}} \\ &\lesssim \sum_{k \text{ s.t. } 2^k < \frac{1}{|E|^{1/n(n-1)}}} |E|^{\frac{1}{n-1}} \\ &\lesssim |E|^{\frac{1}{n-1}} \log\left(1/|E|\right). \end{aligned}$$

Combining these gives the upper bound on L.

For the lower bound on L, simply take all $E_j = \mathrm{cap}_{2\times 2^{-k_0}}(u_{\ell^*})$ for some fixed k_0 and ℓ^*. This gives $|E| \sim 2^{k_0(n-1)n}$, and

$$L_2 \geq \sum_{k=1}^{k_0} \left(|E_1| \times \cdots \times |E_n| \right)^{\frac{1}{n-1}} = k_0 |E|^{\frac{1}{n-1}} \gtrsim |E|^{\frac{1}{n-1}} \log\left(1/|E|\right),$$

which establishes the lower bound on L. ■

2.2.5 Whitney decomposition for $n = 2$

We saw in §2.2.3 that the $n = 2$ case is trivial, but we shall now consider an alternative proof which may be more readily adapted to $n \geq 3$.

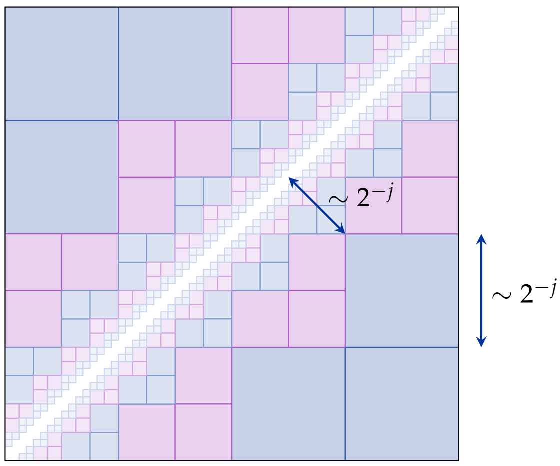
Proposition 2.11 suggests that we need a decomposition which gives us more detailed information about the size of $\omega_1 \wedge \cdots \wedge \omega_n$. For the $n = 2$ case, the "flattened-out" version of the problem, (2.11), with $f_1 = \chi_{E_1}, f_2 = \chi_{E_2}$ ($E_1, E_2 \subseteq [0,1]$) becomes

$$\int_{\mathbb{R}^2} \left(\int_{E_1} \int_{E_2} \chi_{T_{\omega_1}}(x) \chi_{T_{\omega_2}}(x) \omega_1 \wedge \omega_2 \, d\omega_1 d\omega_2 \right) dx \lesssim |E|. \tag{2.15}$$

Now by the same steps which led to (2.14), the left-hand side can be bounded by

$$\sum_{k=1}^{\infty} 2^k \sum_{\ell} \left(\iint_{E \cap (\mathrm{cap}_{2 \times 2^{-k}}(u_\ell))^2} \omega_1 \wedge \omega_2 \, d\omega_1 d\omega_2 \right).$$

Since $\omega_1 \wedge \omega_2 \sim |\omega_1 - \omega_2|$, we consider a Whitney decomposition [Ste70, p16] of each $(\mathrm{cap}_{2^{-k}}(u_\ell))^2$ into squares of sidelength 2^{-j} ($j \geq k$) whose distance to the diagonal is $\sim 2^{-j}$:



Let us denote the union of all the 2^{-j}-squares in this covering by $A_{j,k}(u_\ell)$. Using the fact that on $A_{j,k}(u_\ell)$ we have $\omega_1 \wedge \omega_2 \lesssim 2^{-j}$,

$$\text{LHS (2.15)} \lesssim \sum_{k=1}^{\infty} 2^k \sum_{\ell} \left(\sum_{j \geq k} \iint_{E \cap A_{j,k}(u_\ell)} d\omega_1 d\omega_2 \, 2^{-j} \right).$$

Now because there is no power on this bracket ($\frac{1}{2-1} = 1$) we can move the ℓ sum inside, giving

$$\text{LHS (2.15)} \lesssim \sum_{k=1}^{\infty} 2^k \sum_{j \geq k} 2^{-j} \sum_{\ell} |E \cap A_{j,k}(u_\ell)|.$$

Now if A_j is the set of (ω_1, ω_2) which are $\sim 2^{-j}$ from the diagonal, we have that

$$\sum_{\ell} |E \cap A_{j,k}(u_\ell)| \sim |E \cap A_j|$$

independently of k. Thus

$$\text{LHS (2.15)} \lesssim \sum_{j=1}^{\infty} 2^{-j} \sum_{k \leq j} 2^k |E \cap A_j| \lesssim 2^{-j} 2^j \sum_{j=1}^{\infty} |E \cap A_j| \lesssim |E|.$$

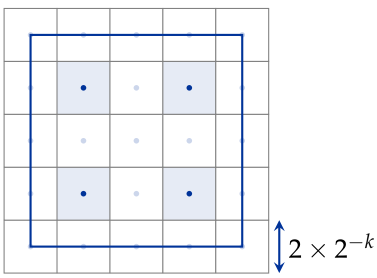
2.3 Main result

We now aim to adapt the idea of the Whitney decomposition to address the more general inequality (2.9) for $n > 2$.

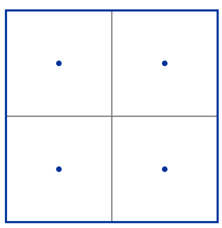
Proceeding as in (2.14) before flattening out as in (2.11), we reduce the question to showing that

$$\sum_{k=2}^{\infty} 2^k \sum_{\ell} \left(\int \cdots \int_{(D_\ell)^n} f_1(\omega_1) \cdots f_n(\omega_n) \omega_1 \wedge \cdots \wedge \omega_n \, d\omega_1 \cdots d\omega_n \right)^{\frac{1}{n-1}}$$
$$\lesssim \prod_{j=1}^{n} \left(\int_{[0,1]^{n-1}} f_j(\omega) \, d\omega \right)^{\frac{1}{n-1}}, \quad (2.16)$$

where, for each k, the sum in ℓ is over the lattice of 2×2^{-k}-separated points in $[0,1]^{n-1}$, and D_ℓ denotes the cube of sidelength 4×2^{-k} centred at ℓ in $[0,1]^{n-1}$. Note that the D_ℓ take the role of the "caps" (cf. (2.14)). We also break up the ℓ sum into 2^{n-1} groups, so that the D_ℓ are disjoint in each group. For instance, when $n = 3$ we separate the ℓ's into four groups, one of which is illustrated.



Selected ℓ's



Resulting D_ℓ's

We now seek to make a further decomposition of the integral over $(D_\ell)^n$, in terms of $\omega_1 \wedge \cdots \wedge \omega_n$. Let

$$W_j = \left\{(\omega_1, \ldots, \omega_n) \,:\, \omega_1 \wedge \cdots \wedge \omega_n \sim 2^{-j(n-1)}\right\}$$

and put $W_j(\ell) = W_j \cap (D_\ell)^n$, so that

$$\text{LHS (2.16)} \lesssim \sum_{k=2}^{\infty} 2^k \sum_{\ell} \left(\sum_{j \geq k} \int \cdots \int_{W_j(\ell)} f_1(\omega_1) \cdots f_n(\omega_n)\, d\omega_1 \cdots d\omega_n 2^{-j(n-1)} \right)^{\frac{1}{n-1}} .$$

This can be broken up as

$$\text{LHS (2.16)} \lesssim \sum_{k=2}^{\infty} \sum_{\ell} \left(\int \cdots \int_{W_k(\ell)} f_1(\omega_1) \cdots f_n(\omega_n)\, d\omega_1 \cdots d\omega_n \right)^{\frac{1}{n-1}}$$
$$+ \sum_{k=2}^{\infty} 2^k \sum_{\ell} \left(\sum_{j > k} \int \cdots \int_{W_j(\ell)} f_1(\omega_1) \cdots f_n(\omega_n)\, d\omega_1 \cdots d\omega_n 2^{-j(n-1)} \right)^{\frac{1}{n-1}} .$$

We are not yet able to deal with the second sum, but for the first we can make some progress when $n = 3$ by decomposing $W_k(\ell)$ even further. We give the details of this decomposition in §2.3.1, then use it in §2.3.2 to establish our main result:

Theorem 2.12. *For the $n = 3$ case,*

$$\sum_{k=2}^{\infty} \sum_{\ell} \left(\iiint_{W_k(\ell)} f_1(\omega_1) f_2(\omega_2) f_3(\omega_3)\, d\omega_1 d\omega_2 d\omega_3 \right)^{\frac{1}{2}}$$
$$\lesssim \prod_{j=1}^{3} \left(\int_{[0,1]^2} f_j(\omega)\, d\omega \right)^{\frac{1}{2}} . \qquad (2.17)$$

Remark 2.13 (Sharpness of the exponent)**.** We consider replacing the exponents $\frac{1}{2}$ appearing in (2.17) with exponents p — let (2.17)$'$ stand for this modified inequality. Observe that taking each $f_j \equiv 1$, the right-hand side of (2.17)$'$ is simply 1. For the left-hand side,

$$\sum_{k=2}^{\infty} \sum_{\ell} \left(\iiint_{W_k(\ell)} f_1(\omega_1) f_2(\omega_2) f_3(\omega_3)\, d\omega_1 d\omega_2 d\omega_3 \right)^{p} = \sum_{k=2}^{\infty} \sum_{\ell} \left(|W_k(\ell)|\right)^{p} ,$$

and we have

- $\#\ell \sim 2^{2k}$ since the ℓ's are $\sim 2^{-k}$-separated lattice points in $[0,1]^2$,
- $|W_k(\ell)| \gtrsim 2^{-6k}$ since taking the ω_i in 2^{-k}-squares at different corners of D_ℓ gives a subset of $W_k(\ell)$.

So with $f_j \equiv 1$,

$$\text{LHS (2.17)}' \gtrsim \sum_{k=2}^{\infty} 2^{2k} 2^{-6kp} = \sum_{k=2}^{\infty} 2^{2k(1-3p)},$$

which is finite only if $p > \frac{1}{3}$.

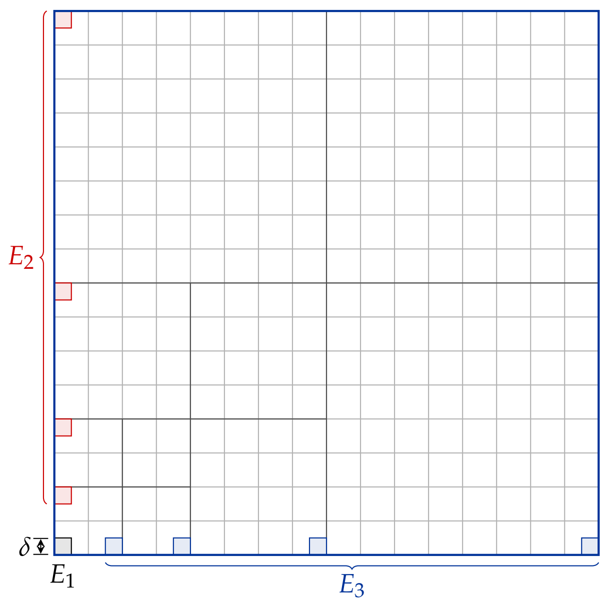
However, the following example shows that $p \geq \frac{1}{2}$ is in fact necessary in order to obtain (2.17)$'$.

The idea is to put $f_j = \chi_{E_j}$ for some sets E_j so that $E = E_1 \times E_2 \times E_3$ is made up of "small" boxes which are chosen in order to obtain a contribution for "many" different k. We will now make this precise.

Given (small) $\delta > 0$, note that $2^{-k} \geq \delta$ when $k \leq \lg \frac{1}{\delta}$. Put

$$D_\delta = \bigcup_{2 \leq k \leq \lg \frac{1}{\delta}} [4 \times 2^{-k} - \delta, 4 \times 2^{-k}]$$

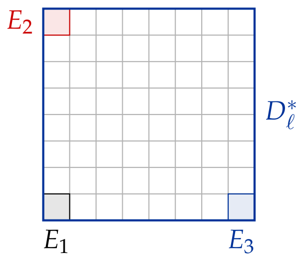
and set $E_1 = [0,\delta]^2$, $E_2 = [0,\delta] \times D_\delta$, $E_3 = D_\delta \times [0,\delta]$, so that E is composed of $\left(\lg \frac{1}{\delta}\right)^2$ δ-boxes. An example of E_1, E_2, E_3 is illustrated in the diagram.



Now, the right-hand side of (2.17)$'$ is

$$\prod_{j=1}^{3}\left(\int_{[0,1]^2} f_j(\omega)\,d\omega\right)^p = \left(\delta^2\left(\lg\tfrac{1}{\delta}\delta^2\right)\left(\lg\tfrac{1}{\delta}\delta^2\right)\right)^p = \left(\lg\tfrac{1}{\delta}\right)^{2p}\delta^{6p},$$

while on the left-hand side we see that, for each k, only the bottom-left D_ℓ (call this D_ℓ^*) can give a contribution, since it is is the only one to overlap the support of f_1. Moreover, in D_ℓ^* only one $\delta \times \delta$ square in the support of each of f_2 and f_3 will contribute — the ones furthest from the support of f_1, since otherwise the value of $\omega_1 \wedge \omega_2 \wedge \omega_3$ is too small to be possible in $W_k(\ell)$. An example of D_ℓ^* with the contributing pieces of the E_i is shown in the diagram.



Hence we obtain one contribution for each $k \le \lg\frac{1}{\delta}$, meaning the left-hand side is

$$\sum_{2\le k\le \lg\frac{1}{\delta}} \delta^{6p} \gtrsim \lg\tfrac{1}{\delta}\delta^{6p}.$$

Thus to obtain a result of the form in Theorem 2.12, we require

$$\lg\tfrac{1}{\delta} \lesssim \left(\lg\tfrac{1}{\delta}\right)^{2p},$$

which is only possible for all $\delta > 0$ if $p \ge \frac{1}{2}$. $\diamondsuit$

Our argument will rely heavily on the following simple observation.

Lemma 2.14. *For $x, a, a', b, b', c, c' \ge 0$,*

$$\sqrt{(x+a)bc} + \sqrt{(x+a')b'c'} \le \sqrt{(x+a+a')(b+b')(c+c')}. \tag{2.18}$$

Proof. Since $\sqrt{x+a}, \sqrt{x+a'} \le \sqrt{x+a+a'}$, it suffices to show

$$\sqrt{bc} + \sqrt{b'c'} \le \sqrt{(b+b')(c+c')},$$

but this is clearly true by applying Cauchy-Schwarz. ■

Remark 2.15. The exponent $\frac{1}{2}$ in (2.18) is sharp, since taking $a = a' = 0$ we would have

$$x^p\left((bc)^p + (b'c')^p\right) \leq x^p\left((b+b')(c+c')\right)^p,$$

but taking $b = b' = c = c' = N$ this gives

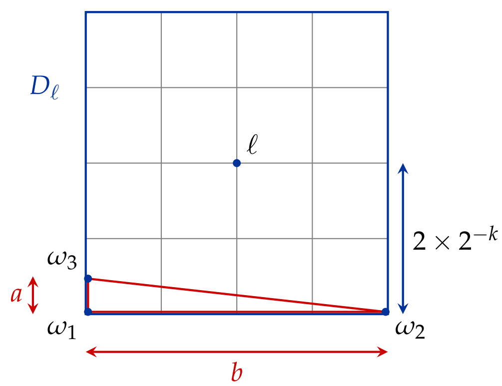
$$2N^p \leq 4^p N^p,$$

from which we see $p \geq \frac{1}{2}$ is required. ◊

2.3.1 Decomposing the main term

Taking the idea of the Whitney decomposition, our aim is to produce a covering of $W_j(\ell)$ by cubes in $[0,1]^{n-1}$ of a certain size, so that we know the wedge does not vary too much on each cube. We have only been able to achieve this successfully for the $n = 3$ case and with $j = k$; in this case, we will show that cubes of sidelength $\sim 2^{-j}$ can be found so that $\omega_1 \wedge \omega_2 \wedge \omega_3 \sim 2^{-2j}$ on each cube.

Remark 2.16. For the $n = 3$ case with general j, it seems that we would need to use cubes of sidelength 2^{k-2j} to ensure the wedge stays $\sim 2^{-2j}$, as illustrated by the following example.



Since $b \sim 2^{-k}$, then $\omega_1 \wedge \omega_2 \wedge \omega_3 \sim 2^{-2j}$ gives

$$ab \sim 2^{-2j} \Leftrightarrow a \sim 2^{k-2j}.$$

So if the ω_i can vary by more than $\sim 2^{k-2j}$, then we could move ω_1 and ω_3 together and make $\omega_1 \wedge \omega_2 \wedge \omega_3$ arbitrarily small. ◊

We shall produce a covering of

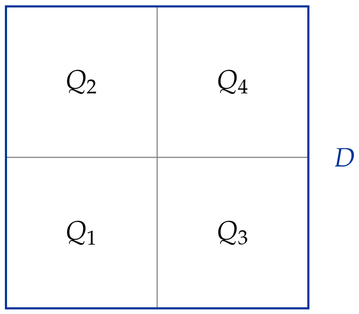
$$W_k(\ell) = \left\{(\omega_1, \omega_2, \omega_3) \,:\, \omega_1 \wedge \omega_2 \wedge \omega_3 \sim 2^{-2k} \text{ and } w_i \in D_\ell\right\},$$

where for concreteness we suppose $\sim$ means

$$\omega_1 \wedge \omega_2 \wedge \omega_3 \leq 16 \times 2^{-2k} \quad \text{and} \quad \omega_1 \wedge \omega_2 \wedge \omega_3 \geq 4 \times 2^{-2k}.$$

Remark 2.17. The constants are chosen here so that, as k varies, the intervals $[4 \times 2^{-2k}, 16 \times 2^{-2k}]$ will cover $[0, 1]$. $\Diamond$

In what follows, if $D \subseteq [0, 1]^2$ is a square then we shall refer to its *quadrants*, meaning the four sub-squares Q_1, Q_2, Q_3, Q_4 as indicated in the following diagram.



Our covering begins with the observation that if all ω_i lie in the same quadrant of D_ℓ, then we have $\omega_1 \wedge \omega_2 \wedge \omega_3 \leq 2 \times 2^{-2k}$ so this triple does not arise in $W_k(\ell)$. Thus to cover $W_k(\ell)$, we need to account for all triples $(\omega_1, \omega_2, \omega_3)$ where each ω_i lies in a different quadrant of D_ℓ, as well as those where exactly two ω_i lie in the same quadrant of D_ℓ. These will be dealt with by "large patterns" and "fine patterns" respectively.

Large patterns

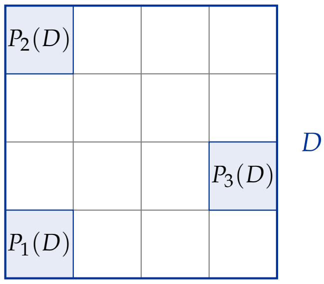
Consider the square $[0, 1]^2$ broken up into a 4×4 grid of sub-squares.

Definition 2.18 (Large patterns)**.** A large pattern is a choice of three squares from the 4×4 grid, each in a different quadrant of $[0, 1]^2$, and with $\omega_1 \wedge \omega_2 \wedge \omega_3 \geq \frac{1}{4}$ for some points ω_i in the interiors of the respective squares.

We write $\mathcal{P}_L$ for the set of large patterns, and note that each $P \in \mathcal{P}_L$ is given by three maps P_i which, given $[0, 1]^2$, return the ith square of the pattern (i.e. the square in which w_i lies).

We extend the definition of the P_i to accept any square $D \subseteq [0,1]^2$ as an input; the corresponding output is the subsquare of D which is obtained by scaling D up to $[0,1]^2$, applying the original P_i, then scaling back down to D. $\Diamond$

An example of a large pattern is shown in the diagram; the regions $P_i(D)$ indicate the outputs of the maps P_i given the input D.



Now, for any $(\omega_1, \omega_2, \omega_3) \in W_k(\ell)$ with each w_i in a different quadrant of D_ℓ, we have $\omega_1 \wedge \omega_2 \wedge \omega_3 \geq 4 \times 2^{-2k}$ from the definition of $W_k(\ell)$. Noting that the area of D_ℓ is 16×2^{-2k}, we see that this means there is a $P \in \mathcal{P}_L$ so that each $\omega_i \in P_i(D_\ell)$ (provided none of the ω_i lie on a boundary of one of the sub-squares of D_ℓ). Thus, up to a set of measure zero,

$$\bigcup_{P \in \mathcal{P}_L} P_1(D_\ell) \times P_2(D_\ell) \times P_3(D_\ell)$$

covers the set of $(\omega_1, \omega_2, \omega_3) \in W_k(\ell)$ with each w_i in a different quadrant of D_ℓ.

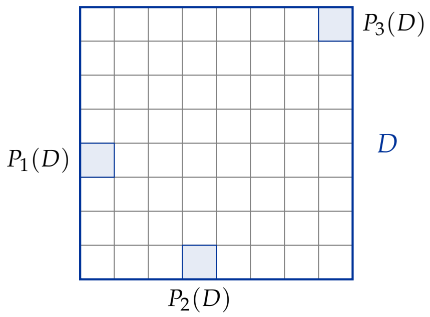
Fine patterns

We now break up $[0,1]^2$ into an 8×8 grid of squares.

Definition 2.19 (Fine patterns)**.** A fine pattern is a choice of three squares from the 8×8 grid such that exactly two are in the same quadrant, and with $\omega_1 \wedge \omega_2 \wedge \omega_3 \geq \frac{1}{4}$ for some points ω_i in the interiors of the respective squares.

We write $\mathcal{P}_F$ for the set of fine patterns. Just as with the large patterns, each $P \in \mathcal{P}_F$ defines three maps P_i which, given a square $D \in [0,1]^2$, return a particular sub-square of D. $\Diamond$

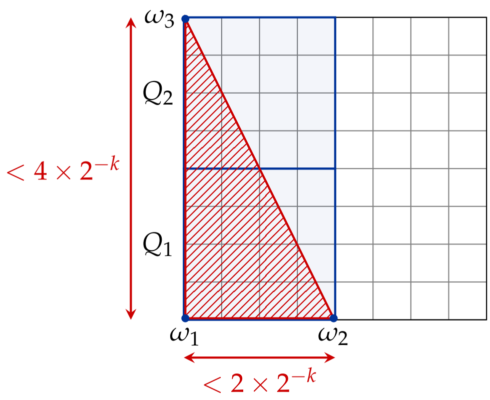
The following diagram shows an example of a fine pattern, acting on the square D.



Remark 2.20. If the quadrant with exactly two squares is Q_1, then we can suppose the third square lies in Q_4, since if the third were in Q_2 or Q_3, we would have

$$\omega_1 \wedge \omega_2 \wedge \omega_3 < \frac{1}{2}(2 \times 2^{-k})(4 \times 2^{-k}) = 4 \times 2^{-2k},$$

as can be seen from the diagram.



$\Diamond$

Just as with the large patterns we see that, up to a set of measure zero,

$$\bigcup_{P \in \mathcal{P}_F} P_1(D_\ell) \times P_2(D_\ell) \times P_3(D_\ell)$$

covers the set of $(\omega_1, \omega_2, \omega_3) \in W_k(\ell)$ with exactly two w_i in the same quadrant of D_ℓ.

This completes our covering of $W_k(\ell)$; up to a set of measure zero, we have

$$W_k(\ell) \subset \bigcup_{P \in \mathcal{P}_L \cup \mathcal{P}_F} P_1(D_\ell) \times P_2(D_\ell) \times P_3(D_\ell).$$

From this we have

$$\iiint\limits_{W_k(\ell)} f_1(\omega_1)f_2(\omega_2)f_3(\omega_3)\,d\omega_1 d\omega_2 d\omega_3$$
$$\le \sum_{P\in\mathcal{P}_L\cup\mathcal{P}_F} \int_{P_1(D_\ell)} f_1(\omega)\,d\omega \int_{P_2(D_\ell)} f_2(\omega)\,d\omega \int_{P_3(D_\ell)} f_3(\omega)\,d\omega,$$

thus our main term is

$$\sum_{k=2}^{\infty}\sum_{\ell}\left(\iiint\limits_{W_k(\ell)} f_1(\omega_1)f_2(\omega_2)f_3(\omega_3)\,d\omega_1 d\omega_2 d\omega_3\right)^{\frac{1}{2}}$$
$$\le \sum_{P\in\mathcal{P}_L\cup\mathcal{P}_F}\sum_{k=2}^{\infty}\sum_{\ell}\left(\int_{P_1(D_\ell)} f_1(\omega)\,d\omega \int_{P_2(D_\ell)} f_2(\omega)\,d\omega \int_{P_3(D_\ell)} f_3(\omega)\,d\omega\right)^{\frac{1}{2}}, \quad (2.19)$$

using the fact that $\|\cdot\|_{\ell^2} \le \|\cdot\|_{\ell^1}$ to bring out the sum over P.

Remark 2.21. It is clear that there are $O(1)$ patterns of each kind, but we can be more precise.

For the large patterns, note that it suffices to consider those with $P_i(D_\ell) \subset Q_i$ for each $i = 1, 2, 3$, since the 24 permutations of these quadrants then produce all possible large patterns. A Maple calculation (Appendix A.1.1) shows that among the 64 possibilities in this reduced class, there are 39 which satisfy the wedge condition in the definition of large patterns. Thus

$$\#P_L = 24 \times 39 = 936.$$

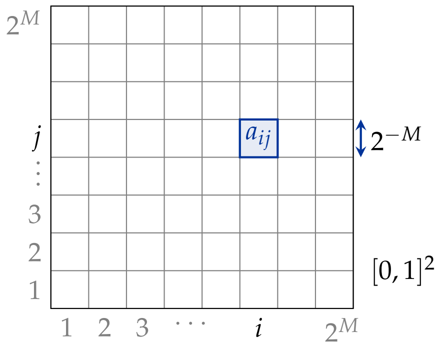
For the fine patterns, note that by Remark 2.20 it suffices to consider the case where two squares lie in Q_1 and the third in Q_4; by rotation there are four times as many fine patterns in total. Another Maple calculation (Appendix A.1.2) shows that 154 of the 4096 possiblities have the required condition on the wedge, so

$$\#P_F = 4 \times 154 = 616. \qquad \Diamond$$

2.3.2 Obtaining a bound for the main term

Now suppose f_1, f_2, f_3 are constant at scale M, in the sense that f_i is constant on each square in the lattice of $2^{-M} \times 2^{-M}$ squares covering $[0,1]^2$. By Lemma 2.8, it is sufficient to consider such functions.

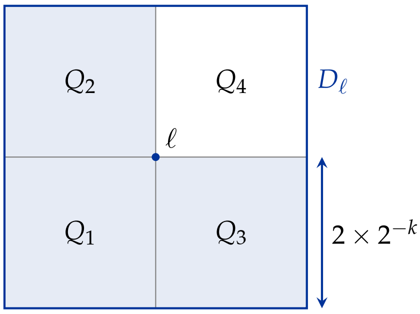
Thus we can suppose f_1 takes the value $a_{ij} \ge 0$ on the square in position (i, j) as shown in the following diagram, and similarly f_2, f_3 take values b_{ij}, c_{ij}.



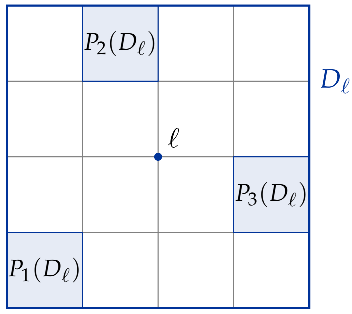
We consider the contributions from $\mathcal{P}_L$ and $\mathcal{P}_F$ in (2.19) as two different cases.

Large patterns

We may suppose (without loss of generality) that the $P_i(D_\ell)$ are in quadrants Q_1, Q_2 and Q_3 of D_ℓ as indicated in the diagram.



e.g.



Lemma 2.22. *For $P \in \mathcal{P}_L$ and f_i constant on 2^{-M}-squares,*

$$\sum_{k=2}^{\infty} \sum_{\ell} \left(\int_{P_1(D_\ell)} f_1(\omega)\, d\omega \int_{P_2(D_\ell)} f_2(\omega)\, d\omega \int_{P_3(D_\ell)} f_3(\omega)\, d\omega \right)^{\frac{1}{2}} \\ \leq \left(\int_{[0,1]^2} f_1(\omega)\, d\omega \int_{[0,1]^2} f_2(\omega)\, d\omega \int_{[0,1]^2} f_3(\omega)\, d\omega \right)^{\frac{1}{2}}. \tag{2.20}$$

Proof (by induction on M). When $M = 0$, the functions are constant on $[0,1]^2$; suppose $f_1 \equiv a$, $f_2 \equiv b$, $f_3 \equiv c$. Then

$$\begin{aligned} \text{LHS (2.20)} &= \sum_{k=2}^{\infty} \sum_{\ell} \sqrt{|P_1(D_\ell)||P_2(D_\ell)||P_3(D_\ell)|abc} \\ &= \sum_{k=2}^{\infty} \sum_{\ell} 2^{-3k} \sqrt{abc}. \end{aligned}$$

For a given k, there are $\left(\frac{1}{4\times 2^{-k}}\right)^2 = 2^{2k-4}$ terms in the ℓ sum, so we have

$$\sum_{k=2}^{\infty} 2^{-k-4}\sqrt{abc} = \tfrac{1}{32}\sqrt{abc},$$

while the right-hand side is simply $\sqrt{abc}$, so we have the result in this case.

For the induction, let us assume (2.20) holds for functions constant on $2^{-(M-1)}$-squares, with constant $C_{M-1} \leq 1$ on the right-hand side. We now suppose the f_i are constant on 2^{-M}-squares. Separating out the first term of the sum in k and grouping the remaining terms, the left-hand side of (2.20) is

$$\begin{aligned}
&\sum_{k=2}^{\infty}\sum_{\ell}\left(\int_{P_1(D_\ell)} f_1(\omega)\,d\omega \int_{P_2(D_\ell)} f_2(\omega)\,d\omega \int_{P_3(D_\ell)} f_3(\omega)\,d\omega\right)^{\frac{1}{2}} \\
&\quad = \prod_{i=1}^{3}\left(\int_{P_i([0,1]^2)} f_i(\omega)\,d\omega\right)^{\frac{1}{2}} + \sum_{Q}\sum_{k=3}^{\infty}\sum_{\substack{\ell \\ \text{s.t. } D_\ell\subseteq Q}}\prod_{i=1}^{3}\left(\int_{P_i(D_\ell)} f_i(\omega)\,d\omega\right)^{\frac{1}{2}}, \qquad (2.21)
\end{aligned}$$

where the sum in Q is over the four quadrants of $[0,1]^2$. Note that this grouping is possible since the D_ℓ always lie entirely in one of the four quadrants of $[0,1]^2$ (thanks to the separation of the sum in ℓ introduced after (2.16)).

Now for each Q we can rescale and obtain

$$\sum_{k=3}^{\infty}\sum_{\substack{\ell \\ \text{s.t. } D_\ell\subseteq Q}}\prod_{i=1}^{3}\left(\int_{P_i(D_\ell)} f_i(\omega)\,d\omega\right)^{\frac{1}{2}} = \sum_{k=2}^{\infty}\sum_{\ell}\prod_{i=1}^{3}\left(\left(\tfrac{1}{2}\right)^2\int_{P_i(D_\ell)} f_i^{(Q)}(\omega)\,d\omega\right)^{\frac{1}{2}},$$

where $f_i^{(Q)} = f_i \circ S_Q$ and $S_Q : [0,1]^2 \to Q$ simply scales $[0,1]^2$ down onto the quadrant Q (so the $f_i^{(Q)}$ are the same as $f_i|_Q$ but scaled up to have domain $[0,1]^2$). In particular, the $f_i^{(Q)}$ are constant on $2^{-(M-1)}$-squares, so we can apply our inductive hypothesis (that (2.20) holds for such functions, with constant C_{M-1} on the right-hand side). For each quadrant Q, this gives

$$\begin{aligned}
&\tfrac{1}{8}\sum_{k=2}^{\infty}\sum_{\ell}\left(\int_{P_1(D_\ell)} f_1^{(Q)}(\omega)\,d\omega \int_{P_2(D_\ell)} f_2^{(Q)}(\omega)\,d\omega \int_{P_3(D_\ell)} f_3^{(Q)}(\omega)\,d\omega\right)^{\frac{1}{2}} \\
&\leq \tfrac{1}{8}C_{M-1}\left(\int_{[0,1]^2} f_1^{(Q)}(\omega)\,d\omega \int_{[0,1]^2} f_2^{(Q)}(\omega)\,d\omega \int_{[0,1]^2} f_3^{(Q)}(\omega)\,d\omega\right)^{\frac{1}{2}} \\
&= C_{M-1}\left(\int_Q f_1(\omega)\,d\omega \int_Q f_2(\omega)\,d\omega \int_Q f_3(\omega)\,d\omega\right)^{\frac{1}{2}}.
\end{aligned}$$

Thus we find that (2.21) is bounded by

$$\left(\int_{P_1([0,1]^2)} f_1(\omega)\,d\omega \int_{P_2([0,1]^2)} f_2(\omega)\,d\omega \int_{P_3([0,1]^2)} f_3(\omega)\,d\omega\right)^{\frac{1}{2}} \\ + C_{M-1}\sum_{Q}\left(\int_{Q} f_1(\omega)\,d\omega \int_{Q} f_2(\omega)\,d\omega \int_{Q} f_3(\omega)\,d\omega\right)^{\frac{1}{2}}. \quad (2.22)$$

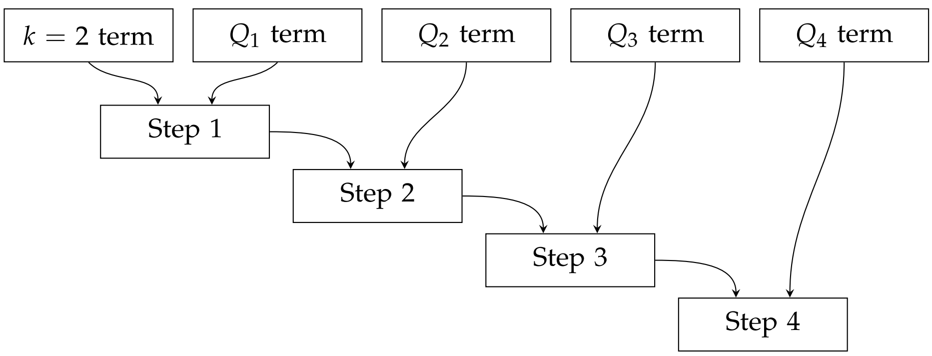
Now, for each Q, no more than one of the $P_i([0,1]^2)$ overlaps it (since a large pattern has at most one square in each quadrant). So in (2.22), each term of the sum in Q can in turn be combined with the first term, using (2.18); for instance, since we can assume $P_1([0,1]^2) \subset Q_1$,

$$\begin{aligned}&\left(\int_{P_1([0,1]^2)} f_1(\omega)\,d\omega \int_{P_2([0,1]^2)} f_2(\omega)\,d\omega \int_{P_3([0,1]^2)} f_3(\omega)\,d\omega\right)^{\frac{1}{2}} \\ &+ C_{M-1}\left(\int_{Q_1} f_1(\omega)\,d\omega \int_{Q_1} f_2(\omega)\,d\omega \int_{Q_1} f_3(\omega)\,d\omega\right)^{\frac{1}{2}} \\ &\leq \left(\int_{Q_1} f_1(\omega)\,d\omega\right)^{\frac{1}{2}}\left(\int_{P_2([0,1]^2)\cup Q_1} f_2(\omega)\,d\omega\right)^{\frac{1}{2}}\left(\int_{P_3([0,1]^2)\cup Q_1} f_3(\omega)\,d\omega\right)^{\frac{1}{2}},\end{aligned}$$

where we have used

$$\begin{aligned}\max\left(\int_{P_1([0,1]^2)} f_1(\omega)\,d\omega, C_{M-1}\int_{Q_1} f_1(\omega)\,d\omega\right) &\leq \max(1, C_{M-1})\int_{Q_1} f_1(\omega)\,d\omega \\ &= \int_{Q_1} f_1(\omega)\,d\omega.\end{aligned}$$

The remaining quadrants' contributions can be combined in the same way, according to the following process (where Step 1 is the calculation above):



The end result of this gives (2.20), which completes the proof by induction. ■

Remark 2.23. Returning to the matter of the sharpness of the exponent $\frac{1}{2}$, as discussed in Remark 2.13, we note that the preceding proof relies on $p = \frac{1}{2}$ when combining the terms in (2.22).

To see this, note that we must bound (2.22) by the right-hand side of (2.20) exactly — i.e. with constant 1. Now using the example from Remark 2.13, with $\delta = 2^{-N}$ for some large $N \in \mathbb{N}$, we find that (2.22) with p in place of $\frac{1}{2}$ is

$$\left(2^{-2N}2^{-2N}2^{-2N}\right)^p + \left(2^{-2N}(N-1)2^{-2N}(N-1)2^{-2N}\right)^p,$$

since only the first term and the $Q = Q_1$ term are nonzero. Since the required bound is

$$\left(2^{-2N}N2^{-2N}N2^{-2N}\right)^p = N^{2p}2^{-6Np},$$

we require

$$1 + (N-1)^{2p} \leq N^{2p},$$

i.e.

$$N^{2p} - (N-1)^{2p} \geq 1.$$

Now by the mean value theorem applied to $g(x) = x^{2p}$, the left-hand side of this inequality is $g'(x_0)$ for some $x_0 \in (N-1, N)$. Thus to have the inequality hold for arbitrarily large N we must have

$$g'(x) = 2px^{2p-1} \geq 1$$

for all large x, which is possible only if $2p - 1 \geq 0$, i.e. $p \geq \frac{1}{2}$. $\lozenge$

Remark 2.24. We also note that in (2.20), the constant 1 on the right-hand side is the best possible. This can be seen by considering the example

$$f_i(\omega) = \chi_{P_i([0,1]^2)}(\omega), \quad i = 1, 2, 3;$$

since only the $k = 2$ term is nonzero on the left-hand side, we find that (2.20) is an equality. $\lozenge$

Fine patterns

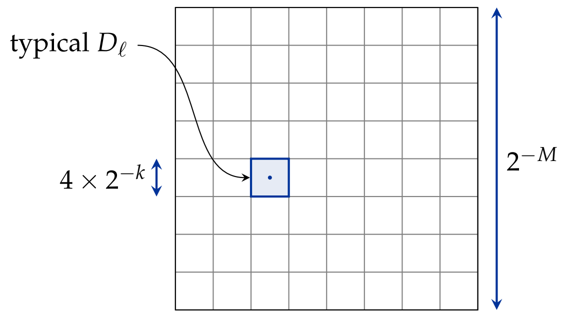
We have seen that it suffices to consider $P \in \mathcal{P}_F$ for which $P_1([0,1]^2), P_2([0,1]^2) \subseteq Q_1$ and $P_3([0,1]^2) \subseteq Q_4$.

We would like to proceed as in the inductive proof for the large patterns, but since P_1 and P_2 now both map to the same quadrant we will not be able to combine terms as in (2.22). Instead, we break the sum into two parts at $k = M+1$, and deal with each part separately.

The tail, $k > M+1$. In this case, the D_ℓ are subsets of the 2^{-M}-squares on which the f_i are constant, so each contribution is of the form

$$\sqrt{|P_1(D_\ell)||P_2(D_\ell)||P_3(D_\ell)|a_{ij}b_{ij}c_{ij}} = 2^{-3(k+1)}\sqrt{a_{ij}b_{ij}c_{ij}}.$$

For each $1 \le i, j \le 2^M$, there are $(2^{k-2-M})^2$ different D_ℓ's for each k.



This gives

$$\begin{aligned}
&\sum_{k>M+1}\sum_{\ell}\left(\int_{P_1(D_\ell)} f_1(\omega)\,d\omega \int_{P_2(D_\ell)} f_2(\omega)\,d\omega \int_{P_3(D_\ell)} f_3(\omega)\,d\omega\right)^{\frac{1}{2}} \\
&= \sum_{i,j=1}^{2^M}\left(\sum_{k>M+1}(2^{k-2-M})^2 2^{-3(k+1)}\right)\sqrt{a_{ij}b_{ij}c_{ij}} \\
&= \tfrac{1}{256}2^{-3M}\sum_{i,j=1}^{2^M}\sqrt{a_{ij}b_{ij}c_{ij}} \\
&\le \tfrac{1}{256}\left(\int_{[0,1]^2} f_1(\omega)\,d\omega \int_{[0,1]^2} f_2(\omega)\,d\omega \int_{[0,1]^2} f_3(\omega)\,d\omega\right)^{\frac{1}{2}}, \qquad (2.23)
\end{aligned}$$

where (2.5) was applied in the last step.

The head, $k \le M+1$. These terms can be dealt with in a similar way to the inductive proof for large patterns, but the argument is slightly more complicated and requires strong induction.

Lemma 2.25. *For $P \in \mathcal{P}_F$ and f_i constant on 2^{-M}-squares,*

$$\begin{aligned} L_M(F) := \sum_{k=2}^{M+1} \sum_{\ell} \left(\int_{P_1(D_\ell)} f_1(\omega)\, d\omega \int_{P_2(D_\ell)} f_2(\omega)\, d\omega \int_{P_3(D_\ell)} f_3(\omega)\, d\omega \right)^{\frac{1}{2}} \\ \leq \left(\int_{[0,1]^2} f_1(\omega)\, d\omega \int_{[0,1]^2} f_2(\omega)\, d\omega \int_{[0,1]^2} f_3(\omega)\, d\omega \right)^{\frac{1}{2}}. \end{aligned} \tag{2.24}$$

Proof (by induction). This is clearly true when $M = 1$, since in that case the left-hand side has only one term.

Now we suppose that (2.24) holds for $L_1, \ldots, L_{M-1}$, and note that

$$L_M(F) = \underbrace{\prod_{i=1}^{3} \left(\int_{P_i([0,1]^2)} f_i(\omega)\, d\omega \right)^{\frac{1}{2}}}_{k=2 \text{ term}} + \tfrac{1}{8} \sum_{Q} L_{M-1}\left(F^{(Q)}\right)$$

where $F^{(Q)} = f_1^{(Q)}, f_2^{(Q)}, f_3^{(Q)}$ (cf. (2.21)).

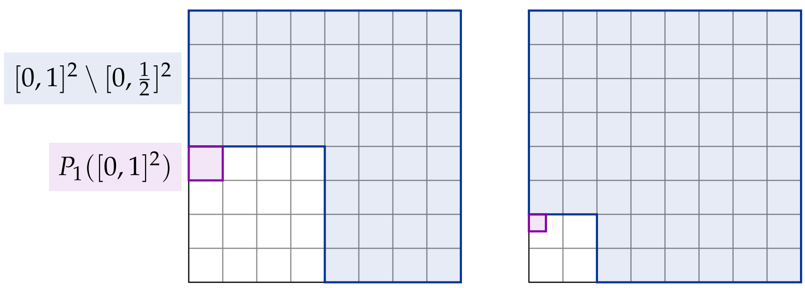
Since the $f_i^{(Q)}$ are constant on $2^{-(M-1)}$-squares, we can apply the inductive hypothesis (2.24) for each $Q \in \{Q_2, Q_3, Q_4\}$ to get

$$\tfrac{1}{8} L_{M-1}\left(F^{(Q)}\right) \leq \left(\int_Q f_1(\omega)\, d\omega \int_Q f_2(\omega)\, d\omega \int_Q f_3(\omega)\, d\omega \right)^{\frac{1}{2}},$$

and these can each be combined with the $k = 2$ term using (2.18), since at most one of the $P_i([0,1]^2)$ will lie in any $Q \in \{Q_2, Q_3, Q_4\}$. The result of this combination is a term bounded by $G_{M-1}(F)$, where for any m

$$G_{M-m}(F) = \prod_{i=1}^{3} \left(\int_{([0,1]^2 \setminus [0,2^{-m}]^2) \cup P_i([0,2\times 2^{-m}]^2)} f_i(\omega)\, d\omega \right)^{\frac{1}{2}}.$$

The regions of integration in the definition of G_{M-m} are illustrated below, for the f_1 term (and a specific choice of pattern).



$m = 1$ $\qquad$ $m = 2$

Thus we have

$$L_M(F) \leq \tfrac{1}{8} L_{M-1}\left(F^{(Q_1)}\right) + G_{M-1}(F).$$

We now repeat this decomposition process on $L_{M-1}\left(F^{(Q_1)}\right)$, obtaining

$$L_{M-1}\left(F^{(Q_1)}\right) = \sum_Q \tfrac{1}{8} L_{M-2}\left(F^{(Q_1)(Q)}\right) + \prod_{i=1}^{3}\left(\int_{P_i([0,1]^2)} f_i^{(Q_1)}(\omega)\, d\omega\right)^{\frac{1}{2}},$$

so

$$\begin{aligned} L_M(F) \leq{}& \tfrac{1}{8^2} L_{M-2}\left(F^{(Q_1)(Q_1)}\right) \\ &+ \sum_{Q \neq Q_1} \tfrac{1}{8^2} L_{M-2}\left(F^{(Q_1)(Q)}\right) + \prod_{i=1}^{3}\left(\int_{P_i(Q_1)} f_i(\omega)\, d\omega\right)^{\frac{1}{2}} + G_{M-1}(F). \end{aligned} \tag{2.25}$$

Now the inductive assumption (2.24) for L_{M-2} can be used on the terms in the sum over $Q \neq Q_1$, and we see that the last three terms in (2.25) can all be combined using (2.18), giving

$$L_M(F) \leq \tfrac{1}{8^2} L_{M-2}\left(F^{(Q_1)(Q_1)}\right) + G_{M-2}(F).$$

Continuing in this way, we arrive at

$$L_M(F) \leq \tfrac{1}{8^{M-1}} L_1\left(F^{(Q_1)\cdots(Q_1)}\right) + G_1(F),$$

where there are $M-1$ exponents Q_1. Now

$$\begin{aligned} \tfrac{1}{8^{M-1}} L_1\left(f_1^{(Q_1)\cdots(Q_1)}, f_2^{(Q_1)\cdots(Q_1)}, f_3^{(Q_1)\cdots(Q_1)}\right) &= \prod_{i=1}^{3}\left(\int_{P_i([0,1]^2)} f_i^{(Q_1)\cdots(Q_1)}(\omega)\, d\omega\right)^{\frac{1}{2}} \\ &= \prod_{i=1}^{3}\left(\int_{P_i([0,2\times 2^{-M}]^2)} f_i(\omega)\, d\omega\right)^{\frac{1}{2}}, \end{aligned}$$

which can be combined with $G_1(F)$ since at most one $P_i([0, 2\times 2^{-M}]^2)$ can overlap the corresponding region of integration in G_1.

This completes the inductive proof of (2.24). ∎

Combining (2.20), (2.23) and (2.24), we obtain

$$\left(\sum_{P\in\mathcal{P}_L} + \sum_{P\in\mathcal{P}_F}\right)\sum_{k=2}^{\infty}\sum_{\ell}\left(\int_{P_1(D_\ell)} f_1(\omega)\,d\omega \int_{P_2(D_\ell)} f_2(\omega)\,d\omega \int_{P_3(D_\ell)} f_3(\omega)\,d\omega\right)^{\frac{1}{2}}$$
$$\leq \left(\#\mathcal{P}_L + \left(1+\tfrac{1}{256}\right)\#\mathcal{P}_F\right)\left(\int_{[0,1]^2} f_1(\omega)\,d\omega \int_{[0,1]^2} f_2(\omega)\,d\omega \int_{[0,1]^2} f_3(\omega)\,d\omega\right)^{\frac{1}{2}},$$

which, in light of (2.19), gives (2.17).

2.4 Aside: Whitney decomposition question

We now establish that the exponent $\frac{1}{n-1}$ is the correct one, in the $n=2$ case.

Theorem 2.26 (Sharp exponent for $n=2$ problem)**.** *Let $E = E_1 \times E_2 \subseteq [0,1]^2$, and let $\mathcal{Q}$ be the set of Whitney cubes for $[0,1]^2 \backslash \{(x,x) : x \in [0,1]\}$.*

(a) For all $p \geq 1$ we have
$$\sum_{Q\in\mathcal{Q}} |E\cap Q|^p \leq C_p |E|^p. \tag{2.26}$$

(b) For $\frac{1}{2} < p < 1$ we have
$$\sum_{Q\in\mathcal{Q}} |E\cap Q|^p \leq C_p \left(\log \tfrac{1}{|E|}\right)|E|^p.$$

(c) For $\frac{1}{2} < p < 1$ there is no C_p which guarantees that (2.26) *holds; some extra dependence on $|E|$ is required.*

Remark 2.27. Taking $E = [0,1]^2$, the left-hand side of (2.26) becomes

$$\sum_{Q\in\mathcal{Q}} |Q|^p \lesssim \sum_k (2^{-2k})^p \times 2^k = \sum_k 2^{(1-2p)k},$$

which is finite only if $1-2p<0$; this shows that $p > \frac{1}{2}$ is necessary. ♢

Proof of Theorem 2.26.

(*a*) For $p = 1$ this is an equality with $C_1 = 1$, as the Q are disjoint. Then for $p > 1$, we use the fact that $\ell^1 \subseteq \ell^p$ to obtain

$$\sum_{Q\in\mathcal{Q}} |E\cap Q|^p \leq \left(\sum_{Q\in\mathcal{Q}} |E\cap Q|\right)^p = C_1^p |E|^p.$$

(*b*) Let $A(k) = \sum_{\substack{Q \in \mathcal{Q} \\ |Q| \sim 2^{-k}}} |E \cap Q|^p$. Then, for each fixed k,

$$\begin{aligned}
A(k) &= \sum_{|Q| \sim 2^{-k}} |E_1 \cap Q^{(1)}|^p |E_2 \cap Q^{(2)}|^p \\
&\overset{\text{Hölder}}{\leq} \left(\sum_{|Q| \sim 2^{-k}} |E_1 \cap Q^{(1)}|^{2p} \right)^{1/2} \left(\sum_{|Q| \sim 2^{-k}} |E_2 \cap Q^{(2)}|^{2p} \right)^{1/2} \\
&\overset{\ell^2 \subseteq \ell^{2p}}{\leq} \left(\sum_{|Q| \sim 2^{-k}} |E_1 \cap Q^{(1)}| \right)^{p} \left(\sum_{|Q| \sim 2^{-k}} |E_2 \cap Q^{(2)}| \right)^{p} \\
&\leq |E|^p,
\end{aligned} \tag{2.27}$$

where our use of $\ell^2 \subseteq \ell^{2p}$ makes this valid for $p \geq \frac{1}{2}$.

For $k \geq \frac{1}{2} \lg \frac{1}{|E|}$, we have $2^{-2k} \leq |E|$, i.e. $|Q| \leq |E|$. Then

$$\begin{aligned}
\sum_{k \geq \frac{1}{2} \lg \frac{1}{|E|}} A(k) &= \sum_{k \geq \frac{1}{2} \lg \frac{1}{|E|}} \sum_{|Q| \sim 2^{-k}} |E \cap Q|^{\frac{1}{2}} \underbrace{|E \cap Q|^{p - \frac{1}{2}}}_{\text{use } |E \cap Q| \leq |Q| \text{ here}} \\
&\leq \sum_{k \geq \frac{1}{2} \lg \frac{1}{|E|}} \underbrace{\sum_{|Q| \sim 2^{-k}} |E \cap Q|^{\frac{1}{2}}}_{\leq |E|^{\frac{1}{2}} \text{ by Hölder}} 2^{-2k(p - \frac{1}{2})} \\
&\leq \frac{2^{-(p - \frac{1}{2}) \lg \frac{1}{|E|}}}{1 - 2^{-2(p - \frac{1}{2})}} |E|^{\frac{1}{2}} \\
&= C_p |E|^{p - \frac{1}{2}} |E|^{\frac{1}{2}} \\
&= C_p |E|^p.
\end{aligned}$$

Now for $k < \frac{1}{2} \lg \frac{1}{|E|}$ we simply use (2.27), giving

$$\begin{aligned}
\sum_{Q \in \mathcal{Q}} |E \cap Q|^p &= \sum_{k < \frac{1}{2} \lg \frac{1}{|E|}} A(k) + \sum_{k \geq \frac{1}{2} \lg \frac{1}{|E|}} A(k) \\
&\leq \left(\frac{1}{2} \lg \frac{1}{|E|} - 1 \right) C_p |E|^p + C_p |E|^p \\
&\leq C_p \frac{1}{2} \lg \frac{1}{|E|} |E|^p.
\end{aligned}$$

■

(*c*) Given a large integer N, put $E_1 = [0, 2^{-N}]$ and take E_2 to be a union of 2^{-N}-intervals, one in each interval $[2^{-(k+1)}, 2^{-k}]$ with $k < N$. Thus E is a union of N boxes, each with area 2^{-2N}. Note that each Whitney cube contains at most one of these small boxes.

Now (2.26) becomes

$$N2^{-2Np} \le C_p \left(N2^{-2N}\right)^p$$

which means we need

$$C_p \ge N^{1-p}$$

for every N. This cannot be achieved with an absolute constant C_p if $p < 1$, showing that the right-hand side must include some extra dependence on $|E|$.

Remark 2.28. The result in (b) is close to being sharp, as can be seen from the example considered in the proof of (c) which shows that the right-hand side must be larger than N^{1-p} where $N2^{-2N} = |E|$. Using the fact that N is a large integer, we have

$$2^{-2N} \le N2^{-2N} = |E|,$$

so $N \ge \frac{1}{2} \lg \frac{1}{|E|}$. Thus the example requires the right-hand side to be at least

$$\left(\frac{1}{2} \lg \frac{1}{|E|}\right)^{1-p},$$

which is close to the result in (b). $\diamondsuit$

2.4.1 Dealing with the small k

For the set of "good k", i.e. $G = \left\{k \le \frac{1}{2} \lg \frac{1}{|E|} \ : \ A(k) \le C\frac{|E|^p}{\lg \frac{1}{|E|}}\right\}$, we have

$$\sum_{k\in G} A(k) \le \underbrace{\#G}_{\le \frac{1}{2} \lg \frac{1}{|E|}} C\frac{|E|^p}{\lg \frac{1}{|E|}} \le \tfrac{1}{2} C|E|^p.$$

Thus we are left with the sum over the "bad k",

$$\sum_{k\in B} A(k) \quad \text{where} \quad B = \left\{k \le \tfrac{1}{2} \lg \frac{1}{|E|} \ : \ A(k) > C\frac{|E|^p}{\lg \frac{1}{|E|}}\right\}.$$

We would be done if $\#B = O(1)$, for then

$$\sum_{k\in B} A(k) \le \#B|E|^p \le C|E|^p$$

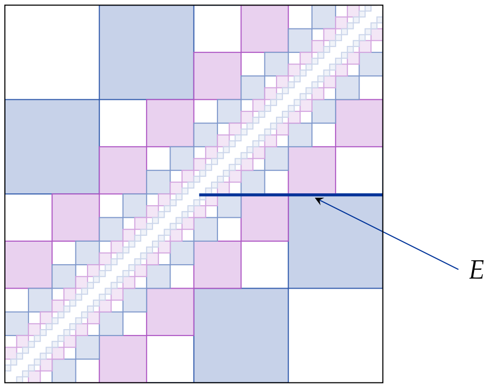
just by applying (2.27) to each $A(k)$.

But the following example shows that we do not have $\#B = O(1)$.

Example 2.29. For any $K \in \mathbb{N}$ put $E = \left[\frac{1}{2} + 2^{-K}, 1\right] \times \left[\frac{1}{2} - \varepsilon, \frac{1}{2}\right]$, for a sufficiently small ε (to be determined later).

Then $B = \{1, 2, \ldots, K\}$.

Proof. An example of E is illustrated in the following diagram — note that for each $k \leq K$ there is only one 2^{-k}-cube which intersects E.



We have $|E| = \varepsilon \times \sum_{k=2}^{K} 2^{-k} = \varepsilon(\frac{1}{2} - 2^{-K})$. Thus to ensure K is bad (i.e. $K \in B$), we must have

$$A(K) = (\varepsilon \times 2^{-K})^p > C\frac{|E|^p}{\lg \frac{1}{|E|}} = C\frac{\varepsilon^p(\frac{1}{2} - 2^{-K})^p}{\lg \frac{1}{\varepsilon\left(\frac{1}{2} - 2^{-K}\right)}},$$

thus we require

$$\begin{aligned} 2^{-Kp} \lg \frac{1}{\varepsilon(\frac{1}{2} - 2^{-K})} &> C(\tfrac{1}{2} - 2^{-K})^p \\ \frac{1}{\varepsilon(\frac{1}{2} - 2^{-K})} &> 2^{C(2^{K-1} - 1)^p} \\ \varepsilon &< \frac{2^{-C(2^{K-1}-1)^p}}{\frac{1}{2} - 2^{-K}}. \end{aligned}$$

Now note that, for $k < K$, the condition for k to be "bad" becomes

$$(\varepsilon 2^{-k})^p > C\frac{\varepsilon^p(\frac{1}{2} - 2^{-K})^p}{\lg \frac{1}{\varepsilon\left(\frac{1}{2} - 2^{-K}\right)}}$$

which can be similarly rearranged to give

$$\varepsilon < \frac{2^{-C2^{kp}(\frac{1}{2} - 2^{-K})^p}}{\frac{1}{2} - 2^{-K}}.$$

Since for $k < K$ this condition is less restrictive than the "bad K" condition, we have that all $k \leq K$ are bad, i.e. $B = \{1, 2, \ldots, K\}$. ■

Still, this example is *not* a counterexample to (2.26); we have

$$\text{LHS} = \sum_{k \in B} A(k) = \sum_{k \leq K} A(k) = \sum_{k \leq K} (\varepsilon 2^{-k})^p = \varepsilon^p (2^{-p} - 2^{-Kp}),$$

while the desired right-hand side is $C\varepsilon^p(\frac{1}{2} - 2^{-K})^p$. This means (2.26) reduces to

$$2^{-p} - 2^{-Kp} \leq C(\tfrac{1}{2} - 2^{-K})^p,$$

which is true as 2^{-p} lies between the two sides.

CHAPTER

3 Wolff-type inequalities

In this chapter we look at the endpoint case of the mixed-norm Wolff-type inequality, and in a series of case studies (§3.4–§3.6) we obtain positive results when certain classes of function are used in the inequalities — as well as a result in §3.2 which shows these endpoint inequalities are sharp.

3.1 The Question

Recall from Chapter 1 that we are interested in determining the dependence of A_δ on δ in the mixed-norm Wolff-type inequality

$$\Big\|\sum_j f_j\Big\|_p \le A_\delta \left(\sum_j \|f_j\|_p^2\right)^{1/2}, \tag{3.1}$$

where the f_j have supp $\hat{f}_j \subseteq S_j$ for some "$\delta^{1/2}$-slabs" S_j, given by

$$S_j = \left\{(\xi', \xi_n) \in \Sigma^\delta : |\xi' - y_j| \le C\delta^{1/2}\right\},$$

where

- Σ^δ is the truncated paraboloid in $\mathbb{R}^n$,

$$\Sigma^\delta = \left\{(\xi', \xi_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \left|\xi_n - \tfrac{1}{2}|\xi'|^2\right| \le \delta, |\xi'| \le 1\right\},$$

- the $y_j \in \mathbb{R}^{n-1}$ are $\delta^{1/2}$-separated; typically $y_j = j\delta^{1/2}$ with $j \in \mathbb{Z}^{n-1}$.

Note that with $n = 2$ and $n = 3$ the endpoints are $p = 6$ and $p = 4$ respectively. Since these are even integers, it is possible to "multiply out" the norm, as in [Cór77, p14]. These are the only dimensions for which the endpoint p is an even integer; for this reason, we will focus attention on $n = 2, 3$.

The aim for both of these cases is to establish whether or not (3.1) holds, and to determine how A_δ should depend on δ; as we saw in Chapter 1, the conjecture is that $A_\delta = C_\varepsilon \delta^{-\varepsilon}$ or perhaps $A_\delta = O\left(\left(\log \frac{1}{\delta}\right)^N\right)$ for some N — we may even hope for $A_\delta = O(1)$.

In fact in §3.2 we establish that (3.1) *cannot* hold with $A_\delta = O(1)$.

In light of this, our aim is to establish (3.1) with $A_\delta = C_\varepsilon \delta^{-\varepsilon}$; an interpolation argument should then give the result of [GS10] for the full conjectured range of p.

Since this is a difficult problem, our approach has been to investigate it in the context of a series of case studies; the idea is to probe the inequality systematically for weaknesses, using functions with various special properties.

We investigate the following cases:

§3.4: **Some fine detail on each slab** — we establish a sharp result in this case (see Theorem 3.16), but this relies on an overall structure across the slabs; when the fine details are positioned arbitrarily on each slab, we are unable to obtain the result (see §3.4.2).

§3.5: **Constant on each slab** — we establish the result in the $n = 3$ case.

§3.6: **Every slab with identical fine detail** — we again establish a sharp result.

In the remainder of this section, we clarify the notion of "fine detail", and use this to rewrite the inequality (3.1). There is then a detour in §3.3 to establish some number-theoretic results which are used in the case studies.

3.1.1 Assembling test cases using "blobs"

The idea is to further decompose the slabs S_j into "blobs" of radius δ, and specify the value of $\hat{f}$ on each blob. The blobs will be created using a smooth bump function,

$$\phi \in C_0^\infty, \text{ with } \phi = 1 \text{ on } B(0,1) \text{ and } \phi = 0 \text{ outside } B(0,2).$$

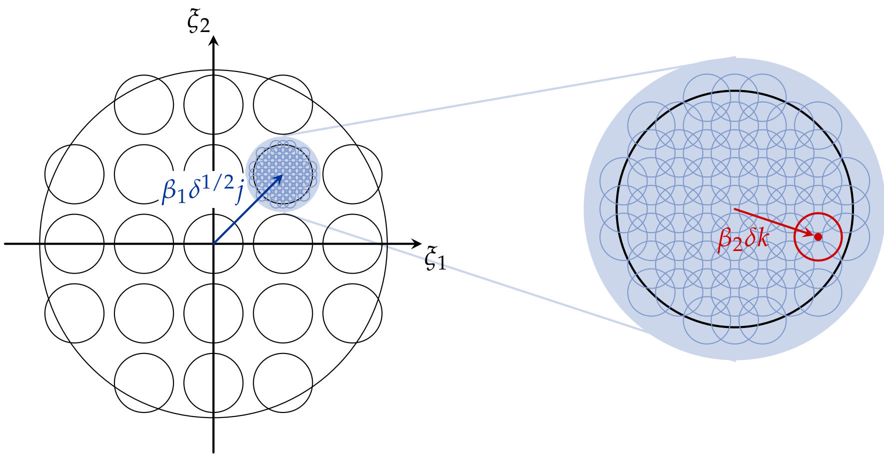
For the centres of the blobs, we use a set of δ-separated points $\{w_{jk}\}_{j,k}$ lying on the paraboloid, with w_{jk} lying in the slab S_j. In fact, we take these points to lie on a lattice; they are given explicitly by the formula

$$w_{jk} = (s_{jk}, \tfrac{1}{2}|s_{jk}|^2), \quad \text{where } s_{jk} = \alpha + \beta_1\delta^{1/2}j + \beta_2\delta k \tag{3.2}$$

with $\alpha \in \mathbb{R}^2$, $j, k \in \mathbb{Z}^2$ and $|j| \leq \frac{1}{\beta_1}\delta^{-1/2}, |k| \leq \frac{1}{\beta_2}\delta^{-1/2}$ (for the $n = 3$ case; for $n = 2$ we take $\alpha \in \mathbb{R}$, $j, k \in \mathbb{Z}$). One can think of s_{jk} like an address; j is the slab 'number' and k is the blob 'number', while α is used to translate the lattice.

Note that $\beta_1\delta^{1/2}, \beta_2\delta > 0$ are real numbers giving the smallest possible distance between the centres of any two slabs and blobs respectively. We also observe that there is a finite range of j and k (there are $\sim \delta^{-(n-1)/2}$ of each).

The following diagram illustrates the positioning of slabs and blobs for the $n = 3$ case, with $\beta_1 = 2, \beta_2 = 1$:



For some choice of constants a_{jk}, we define

$$f_j = \sum_k f_{jk} \quad \text{where} \quad \widehat{f_{jk}}(\xi) = a_{jk}\phi\left(\frac{\xi - w_{jk}}{\delta}\right)\delta^{-n}.$$

From this, we have

$$\begin{aligned} f_{jk}(x) &= \int a_{jk}\phi\left(\frac{\xi - w_{jk}}{\delta}\right)\delta^{-n}e^{2\pi i x\cdot\xi}\,d\xi \\ &= a_{jk}e^{2\pi i x\cdot w_{jk}}\int \phi(u)e^{2\pi i(\delta x)\cdot u}\,du \\ &= a_{jk}e^{2\pi i x\cdot w_{jk}}\check{\phi}(\delta x). \end{aligned}$$

Considering the left-hand side of (3.1), for $n = 3$ we have

$$\begin{aligned}
\Big\|\sum_j f_j\Big\|_4^4 &= \int \Big|\sum_j \sum_k a_{jk} e^{2\pi i x\cdot w_{jk}} \check{\phi}(\delta x)\Big|^4 \, dx \\
&= \int (\sum_{j_1,k_1} \cdots)(\sum_{j_2,k_2} \cdots)\overline{(\sum_{j_3,k_3} \cdots)(\sum_{j_4,k_4} \cdots)}\, dx \\
&= \sum_{\substack{j_1,j_2,j_3,j_4 \\ k_1,k_2,k_3,k_4}} a_{j_1k_1} a_{j_2k_2} \overline{a_{j_3k_3} a_{j_4k_4}} \int (\check{\phi}(\delta x))^4 e^{2\pi i x\cdot(w_{j_1k_1}+w_{j_2k_2}-w_{j_3k_3}-w_{j_4k_4})}\, dx.
\end{aligned}$$

Thus by Fourier inversion,

$$\Big\|\sum_j f_j\Big\|_4^4 = \delta^{-3} \sum_{\substack{j_1,j_2,j_3,j_4 \\ k_1,k_2,k_3,k_4}} a_{j_1k_1} a_{j_2k_2} \overline{a_{j_3k_3} a_{j_4k_4}} \phi^4 \left(\frac{w_{j_1k_1}+w_{j_2k_2}-w_{j_3k_3}-w_{j_4k_4}}{\delta}\right), \tag{3.3}$$

where $\phi^4 = \phi * \phi * \phi * \phi$.

Now considering the right-hand side of (3.1), we can repeat this argument to obtain

$$\begin{aligned}
\|f_j\|_4^4 &= \int \Big|\sum_k a_{jk} e^{2\pi i x\cdot w_{jk}} \check{\phi}(\delta x)\Big|^4 \\
&= \delta^{-3} \sum_{k_1,k_2,k_3,k_4} a_{jk_1} a_{jk_2} \overline{a_{jk_3} a_{jk_4}} \phi^4 \left(\frac{w_{jk_1}+w_{jk_2}-w_{jk_3}-w_{jk_4}}{\delta}\right),
\end{aligned} \tag{3.4}$$

from which we find

$$\begin{aligned}
\mathrm{RHS}^4 &= \left(\sum_j \|f_j\|_4^2\right)^2 \\
&= \delta^{-3} \left(\sum_j \left(\sum_{k_1,k_2,k_3,k_4} a_{jk_1} a_{jk_2} \overline{a_{jk_3} a_{jk_4}} \phi^4 \left(\frac{w_{jk_1}+w_{jk_2}-w_{jk_3}-w_{jk_4}}{\delta}\right)\right)^{1/2}\right)^2.
\end{aligned}$$

Thus for our test case, (3.1) becomes

Question 3.1 ($n = 3$ test case)**.**

$$\begin{aligned}
&\sum_{\substack{j_1,j_2,j_3,j_4 \\ k_1,k_2,k_3,k_4}} a_{j_1k_1} a_{j_2k_2} \overline{a_{j_3k_3} a_{j_4k_4}} \phi^4 \left(\frac{w_{j_1k_1}+w_{j_2k_2}-w_{j_3k_3}-w_{j_4k_4}}{\delta}\right) \\
&\leq A_\delta^4 \left(\sum_j \left(\sum_{k_1,k_2,k_3,k_4} a_{jk_1} a_{jk_2} \overline{a_{jk_3} a_{jk_4}} \phi^4 \left(\frac{w_{jk_1}+w_{jk_2}-w_{jk_3}-w_{jk_4}}{\delta}\right)\right)^{1/2}\right)^2.
\end{aligned} \tag{3.5}$$

The same argument can be carried out for the $n = 2$ case, giving

Question 3.2 ($n = 2$ test case)**.**

$$\sum_{\substack{j_1,\ldots,j_6\\k_1,\ldots,k_6}} a_{j_1k_1}\cdots\overline{a_{j_6k_6}}\phi^6\left(\frac{w_{j_1k_1}+\cdots-w_{j_6k_6}}{\delta}\right)$$
$$\leq A_\delta^6\left(\sum_j\left(\sum_{k_1,\ldots,k_6} a_{jk_1}\cdots\overline{a_{jk_6}}\phi^6\left(\frac{w_{j_1k_1}+\cdots-w_{j_6k_6}}{\delta}\right)\right)^{1/3}\right)^3. \quad (3.6)$$

The following result shows why β_1 is included in (3.2) — if $\beta_1 > 1$ then the slabs are separated, but with the following result we see that this does not lose any generality.

Proposition 3.3 (Separating the slabs)**.** *It is enough to prove the inequality* (3.1) *in the case where the slabs are $\beta\delta^{1/2}$-separated, for some β independent of δ.*

Proof. Suppose we have

$$\Big\|\sum_{j\in J_i} f_j\Big\|_p \leq A_\delta\left(\sum_{j\in J_i}\|f_j\|_p^2\right)^{1/2} \quad (3.7)$$

for each $i \in \Lambda$ where the J_i are a disjoint partition of the js. Then

$$\Big\|\sum_j f_j\Big\|_p \leq \sum_{i\in\Lambda}\Big\|\sum_{j\in J_i} f_j\Big\|_p \overset{(3.7)}{\leq} A_\delta\sum_{i\in\Lambda}\left(\sum_{j\in J_i}\|f_j\|_p^2\right)^{1/2}$$
$$\overset{\text{C-S}}{\leq} A_\delta|\Lambda|^{1/2}\left(\sum_{i\in\Lambda}\sum_{j\in J_i}\|f_j\|_p^2\right)^{1/2}$$

so (3.1) holds with the same δ dependence in the constant, provided $|\Lambda|$ does not depend on δ. For the slabs to be $\beta\delta^{1/2}$-separated, we require only that $|\Lambda|$ depends on β and n. ∎

3.1.2 Simplified test cases: using positive coefficients

If we also assume that the $a_{jk} \geq 0$, we can replace the ϕ terms appearing in (3.5) and (3.6) with conditions on the choice of j_i and k_i, as follows.

Using the fact that $\phi^4 \lesssim \chi_{B(0,8)}$, and that the $a_{jk} \geq 0$, we find that (3.3) gives

$$\mathrm{LHS}^4 \lesssim \delta^{-3} \sum_{\substack{j_1,j_2,j_3,j_4 \\ k_1,k_2,k_3,k_4 \\ \text{s.t. } |w_{j_1k_1}+w_{j_2k_2}-w_{j_3k_3}-w_{j_4k_4}|\leq 8\delta}} a_{j_1k_1}a_{j_2k_2}a_{j_3k_3}a_{j_4k_4}. \tag{3.8}$$

Similarly, using $\phi^4 \geq \chi_{B(0,1)}$ in (3.4) leads to

$$\mathrm{RHS}^4 \geq \delta^{-3} \left(\sum_j \left(\sum_{\substack{k_1,k_2,k_3,k_4 \\ \text{s.t. } |w_{jk_1}+w_{jk_2}-w_{jk_3}-w_{jk_4}|\leq \delta}} a_{jk_1}a_{jk_2}a_{jk_3}a_{jk_4} \right)^{1/2} \right)^2.$$

Thus (3.1) will be true if

$$\begin{aligned}\sum_{\substack{j_1,j_2,j_3,j_4 \\ k_1,k_2,k_3,k_4 \\ \text{s.t. } |w_{j_1k_1}+w_{j_2k_2}-w_{j_3k_3}-w_{j_4k_4}|\leq 2\delta}} & a_{j_1k_1}a_{j_2k_2}a_{j_3k_3}a_{j_4k_4} \\ &\leq A_\delta^4 \left(\sum_j \left(\sum_{\substack{k_1,k_2,k_3,k_4 \\ \text{s.t. } |w_{jk_1}+w_{jk_2}-w_{jk_3}-w_{jk_4}|\leq \delta}} a_{jk_1}a_{jk_2}a_{jk_3}a_{jk_4} \right)^{1/2} \right)^2. \end{aligned} \tag{3.9}$$

By swapping $\chi_{B(0,1)}$ and $\chi_{B(0,2)}$ in this argument, we get reversed bounds for the left- and right-hand sides, and hence (3.1) is true only if

$$\begin{aligned}\sum_{\substack{j_1,j_2,j_3,j_4 \\ k_1,k_2,k_3,k_4 \\ \text{s.t. } |w_{j_1k_1}+w_{j_2k_2}-w_{j_3k_3}-w_{j_4k_4}|\leq \delta}} & a_{j_1k_1}a_{j_2k_2}a_{j_3k_3}a_{j_4k_4} \\ &\leq A_\delta^4 \left(\sum_j \left(\sum_{\substack{k_1,k_2,k_3,k_4 \\ \text{s.t. } |w_{jk_1}+w_{jk_2}-w_{jk_3}-w_{jk_4}|\leq 2\delta}} a_{jk_1}a_{jk_2}a_{jk_3}a_{jk_4} \right)^{1/2} \right)^2. \end{aligned} \tag{3.10}$$

Using separation

To make progress, we will need to have better understanding of when conditions like $|w_{j_1k_1} + w_{j_2k_2} - w_{j_3k_3} - w_{j_4k_4}| \leq \delta$ are satisfied. We shall soon see that it is helpful to use β_2, the blob separation, and the following result justifies doing so.

Proposition 3.4 (Separating the blobs). *If all $a_{jk} \geq 0$ then to establish the endpoint cases of* (3.1) *(n=2,3) we may suppose the blobs are $\beta\delta$-separated, for some $\beta > 8$ independent of δ.*

Proof. Divide all possible ks into disjoint sets K_i, $i \in \Lambda$, so that all the points in K_i are β-separated. Now suppose we have the inequality for each K_i, i.e. for all $i \in \Lambda$

$$\Big\|\sum_j \sum_{k\in K_i} f_{jk}\Big\|_p \leq A_\delta \left(\sum_j \Big\|\sum_{k\in K_i} f_{jk}\Big\|_p^2\right)^{1/2}.$$

Then following the same steps as the proof of Proposition 3.3, we arrive at

$$\Big\|\sum_j \sum_{i\in\Lambda} \sum_{k\in K_i} f_{jk}\Big\|_p \leq A_\delta |\Lambda|^{1/2} \left(\sum_{i\in\Lambda}\sum_j \Big\|\sum_{k\in K_i} f_{jk}\Big\|_p^2\right)^{1/2}$$

which will imply (3.1) if for each j we have

$$\sum_{i\in\Lambda} \Big\|\sum_{k\in K_i} f_{jk}\Big\|_p^2 \lesssim \Big\|\sum_{i\in\Lambda}\sum_{k\in K_i} f_{jk}\Big\|_p^2 = \|f_j\|_p^2. \tag{3.11}$$

Now we restrict attention to the endpoint case; we shall proceed with the $n = 3, p = 4$ case, but the $n = 2, p = 6$ case works in exactly the same way. We use the same argument that gave (3.8) to obtain

$$\begin{aligned}\text{LHS (3.11)} &\leq \sum_{i\in\Lambda}\left(\delta^{-3} \sum_{\substack{k_1,k_2,k_3,k_4\in K_i \\ \text{s.t. } |w_{jk_1}+w_{jk_2}-w_{jk_3}-w_{jk_4}|\leq 8\delta}} a_{jk_1}a_{jk_2}a_{jk_3}a_{jk_4}\right)^{1/2} \\ &\overset{\text{C-S}}{\leq} |\Lambda|^{1/2}\left(\delta^{-3}\sum_{i\in\Lambda} \sum_{\substack{k_1,k_2,k_3,k_4\in K_i \\ \text{s.t. } |w_{jk_1}+w_{jk_2}-w_{jk_3}-w_{jk_4}|\leq 8\delta}} a_{jk_1}a_{jk_2}a_{jk_3}a_{jk_4}\right)^{1/2}.\end{aligned}$$

Also, using the lower bound (3.4) for the right-hand side, we see that (3.11) follows if we have

$$\sum_{i\in\Lambda} \sum_{\substack{k_1,k_2,k_3,k_4\in K_i \\ \text{s.t. } |w_{jk_1}+w_{jk_2}-w_{jk_3}-w_{jk_4}|\le 8\delta}} a_{jk_1}a_{jk_2}a_{jk_3}a_{jk_4} \lesssim \sum_{\substack{k_1,k_2,k_3,k_4 \\ \text{s.t. } |w_{jk_1}+w_{jk_2}-w_{jk_3}-w_{jk_4}|\le \delta}} a_{jk_1}a_{jk_2}a_{jk_3}a_{jk_4} \tag{3.12}$$

Now using (3.2) with $\beta_2 = \beta$ to replace the w_{jk}, the condition on the left-hand side becomes

$$\left|\left(\beta\delta(k_1+k_2-k_3-k_4)\,,\quad \begin{matrix} \beta\delta\alpha\cdot(k_1+k_2-k_3-k_4) \\ +\beta_1\beta\delta^{3/2}j\cdot(k_1+k_2-k_3-k_4) \\ +\frac{1}{2}\beta^2\delta^2(|k_1|^2+|k_2|^2-|k_3|^2-|k_4|^2) \end{matrix}\right)\right| \le 8\delta. \tag{3.13}$$

From the first coordinate, we must have $|k_1+k_2-k_3-k_4| \le \frac{8}{\beta}$. Since $\beta > 8$, this is equivalent to $k_1+k_2 = k_3+k_4$. Plugging this back into (3.13), we get an inequality which is always satisfied since $|k_i| \le \frac{1}{\beta}\delta^{-1/2}$. Using an identical argument for the right-hand side, we see that (3.12) is equivalent to

$$\sum_{i\in\Lambda} \sum_{\substack{k_1,k_2,k_3,k_4\in K_i \\ \text{s.t. } k_1+k_2=k_3+k_4}} a_{jk_1}a_{jk_2}a_{jk_3}a_{jk_4} \lesssim \sum_{\substack{k_1,k_2,k_3,k_4 \\ \text{s.t. } k_1+k_2=k_3+k_4}} a_{jk_1}a_{jk_2}a_{jk_3}a_{jk_4}$$

which is clearly true if all $a_{jk} \ge 0$, since the right-hand side has all the terms of the left-hand side as well as other nonnegative terms. ■

Since we can assume the slabs and blobs are suitably separated, we return to (3.9) and (3.10), and rewrite the right-hand side of each assuming $\beta_2 > 8$.

Thus when all $a_{jk} \ge 0$, (3.1) is true if

$$\sum_{|w_{j_1k_1}+w_{j_2k_2}-w_{j_3k_3}-w_{j_4k_4}|\le 8\delta} a_{j_1k_1}a_{j_2k_2}a_{j_3k_3}a_{j_4k_4} \le A_\delta^4 \left(\sum_j \left(\sum_{k_1+k_2=k_3+k_4} a_{jk_1}a_{jk_2}a_{jk_3}a_{jk_4}\right)^{1/2}\right)^2$$

and only if

$$\sum_{|w_{j_1k_1}+w_{j_2k_2}-w_{j_3k_3}-w_{j_4k_4}|\le \delta} a_{j_1k_1}a_{j_2k_2}a_{j_3k_3}a_{j_4k_4} \le A_\delta^4 \left(\sum_j \left(\sum_{k_1+k_2=k_3+k_4} a_{jk_1}a_{jk_2}a_{jk_3}a_{jk_4}\right)^{1/2}\right)^2.$$

Note that the only difference between these is the δ versus 8δ appearing in the condition on the left-hand side sum. Thus in this simplified setting, (3.1) becomes:

Question 3.5 ($n = 3$ test case (positive coefficients)). *Assuming that all $a_{jk} \geq 0$, do we have*

$$\sum_{|w_{j_1k_1}+w_{j_2k_2}-w_{j_3k_3}-w_{j_4k_4}|\leq c\delta} a_{j_1k_1}a_{j_2k_2}a_{j_3k_3}a_{j_4k_4} \leq A^4_{\delta,c}\left(\sum_j\left(\sum_{k_1+k_2=k_3+k_4} a_{jk_1}a_{jk_2}a_{jk_3}a_{jk_4}\right)^{1/2}\right)^2 \tag{3.14}$$

with $A_{\delta,c} = C_\varepsilon\delta^{-\varepsilon}$? (In particular, when $c = 1, 8$.)

Again, the same arguments can be carried out for $n = 2$, resulting in

Question 3.6 ($n = 2$ test case (positive coefficients)). *Assuming that all $a_{jk} \geq 0$, do we have*

$$\sum_{|w_{j_1k_1}+w_{j_2k_2}+w_{j_3k_3}-w_{j_4k_4}-w_{j_5k_5}-w_{j_6k_6}|\leq c\delta} a_{j_1k_1}a_{j_2k_2}a_{j_3k_3}a_{j_4k_4}a_{j_5k_5}a_{j_6k_6} \leq A^6_{\delta,c}\left(\sum_j\left(\sum_{k_1+k_2+k_3=k_4+k_5+k_6} a_{jk_1}a_{jk_2}a_{jk_3}a_{jk_4}a_{jk_5}a_{jk_6}\right)^{1/3}\right)^3 \tag{3.15}$$

with $A_{\delta,c} = C_\varepsilon\delta^{-\varepsilon}$? (In particular, when $c = 1, 12$.)

Rewriting the condition on the LHS sum ($n = 3$)

On the left-hand side of (3.14), the condition is $|w_{j_1k_1} + w_{j_2k_2} - w_{j_3k_3} - w_{j_4k_4}| \leq c\delta$ which can be written as

$$\left\|\begin{pmatrix} \begin{array}{l} \beta_1\delta^{1/2}(j_1+j_2-j_3-j_4) \\ +\beta_2\delta(k_1+k_2-k_3-k_4) \end{array}, & \begin{array}{l} \beta_1\delta^{1/2}\alpha\cdot(j_1+j_2-j_3-j_4) \\ +\beta_2\delta\alpha\cdot(k_1+k_2-k_3-k_4) \\ +\frac{1}{2}\beta_1^2\delta(|j_1|^2+|j_2|^2-|j_3|^2-|j_4|^2) \\ +\beta_1\beta_2\delta^{3/2}(j_1\cdot k_1+j_2\cdot k_2-j_3\cdot k_3-j_4\cdot k_4) \\ +\frac{1}{2}\beta_2^2\delta^2(|k_1|^2+|k_2|^2-|k_3|^2-|k_4|^2) \end{array} \end{pmatrix}\right\| \leq c\delta. \tag{3.16}$$

Observe that, in particular, the first coordinate must have size $\leq c\delta$; let us denote this as $|J + K| \leq c\delta$. Now since $|K| = |\beta_2\delta(k_1 + \cdots - k_4)| \leq 4\delta^{1/2}$, the other term cannot be too large; more precisely, if $|J| > \frac{9}{2}\delta^{1/2}$ then

$$|J + K| \geq |J| - |K| > \tfrac{1}{2}\delta^{1/2},$$

and $\frac{1}{2}\delta^{1/2} \leq c\delta$ fails for sufficiently small δ. So we must have $\beta_1|j_1 + j_2 - j_3 - j_4| \leq \frac{9}{2}$. Choosing $\beta_1 \geq 5$ gives $j_1 + j_2 = j_3 + j_4$.

The first coordinate now shows that we require $|k_1 + k_2 - k_3 - k_4| \leq \frac{c}{\beta_2}$, but since we assume $\beta_2 > 8$ and $c = 1, 8$ this is simply $k_1 + k_2 = k_3 + k_4$.

The condition (3.16) now reduces to looking at the second coordinate, giving a condition of the form

$$\left|\tfrac{1}{2}\beta_1^2\delta A + \beta_1\beta_2\delta^{3/2}B + \tfrac{1}{2}\beta_2^2\delta^2 C\right| \leq c\delta$$

i.e.

$$\left|\beta_1^2 A + 2\beta_1\beta_2\delta^{1/2}B + \beta_2^2\delta C\right| \leq 2c$$

where

$$\begin{aligned}
A &= |j_1|^2 + |j_2|^2 - |j_3|^2 - |j_4|^2, & |A| &\leq 2\left(\frac{1}{\beta_1}\delta^{-1/2}\right)^2 = \frac{2}{\beta_1^2}\delta^{-1} \\
B &= j_1 \cdot k_1 + j_2 \cdot k_2 - j_3 \cdot k_3 - j_4 \cdot k_4, & |B| &\leq 4\max(|j_i||k_i|) \leq \tfrac{4}{\beta_1\beta_2}\delta^{-1} \\
C &= |k_1|^2 + |k_2|^2 - |k_3|^2 - |k_4|^2, & |C| &\leq \tfrac{2}{\beta_2^2}\delta^{-1}.
\end{aligned}$$

To simplify the presentation, we now fix $\beta_1 = \beta_2 = 10$ so that the condition (3.16) becomes

$$\begin{matrix} j_1 + j_2 = j_3 + j_4 \\ k_1 + k_2 = k_3 + k_4 \end{matrix} \quad \text{and} \quad \left|A + 2\delta^{1/2}B + \delta C\right| \leq \tfrac{c}{50}. \tag{3.17}$$

Using $p = j_1 + j_2$, $m = |j_1|^2 + |j_2|^2$, $q = k_1 + k_2$ we can reindex the sum and write the left-hand side of (3.14) as

$$\underbrace{\sum_p \sum_m \sum_{j_1 \in R_{p,m}} \sum_\gamma \sum_{j_3 \in R_{p,m+\gamma}} \sum_q \sum_{k_1} \sum_{k_3}}_{\text{s.t.}\left|A+2\delta^{1/2}B+\delta C\right| \leq \frac{c}{50}} a_{j_1k_1} a_{(p-j_1)(q-k_1)} a_{j_3k_3} a_{(p-j_3)(q-k_3)} \tag{3.18}$$

where

$$R_{p,m} = \left\{ j : |j|^2 + |p - j|^2 = m \right\}, \tag{3.19}$$

which is the set of integer points on the circle with centre $\frac{1}{2}p$ and radius $\frac{1}{2}\sqrt{2m - |p|^2}$.

Note that m and γ are integers; we certainly have $\frac{1}{2}|p|^2 \leq m \leq \frac{1}{50}\delta^{-1}$, while the range of the γ will be determined later on when necessary.

Remark 3.7 (Maple experiment — see Appendix A.3)**.** In the case $\delta = (12^2)^{-1}$ with $\beta_1 = \beta_2 = 5$ there are 4381 choices of j_i satisfying $j_1 + j_2 = j_3 + j_4$, and also of k_i satisfying $k_1 + k_2 = k_3 + k_4$.

Of these $4381^2 \approx 19$ million choices of j_i and k_i,

- 1 669 521 ($\approx$ 8.70%) satisfy $\left|A + 2\delta^{1/2}B + \delta C\right| \leq \frac{4}{25}$, with $A = 0$;
- 14 656 ($\approx$ 0.08%) satisfy $\left|A + 2\delta^{1/2}B + \delta C\right| \leq \frac{4}{25}$, with $A \neq 0$.

This suggests that $A = 0$ is very much the main term. $\Diamond$

$n = 2$ **version**

The argument in this case is very similar. Note that in the first coordinate, we now have $|K| = |\beta_2\delta(k_1 + \cdots - k_6)| \leq 6\delta^{1/2}$ so the meaning of "too large" is now $|J| > \frac{13}{2}\delta^{1/2}$. Thus if $\beta_1 \geq 7$ we obtain $j_1 + j_2 + j_3 = j_4 + j_5 + j_6$, and as before $\beta_2 > 8$ ensures $k_1 + k_2 + k_3 = k_4 + k_5 + k_6$.

Thus if we fix $\beta_1 = \beta_2 = 10$, the condition on the left-hand side of (3.15) becomes

$$\begin{array}{c} j_1 + j_2 + j_3 = j_4 + j_5 + j_6 \\ k_1 + k_2 + k_3 = k_4 + k_5 + k_6 \end{array} \quad \text{and} \quad \left|A + 2\delta^{1/2}B + \delta C\right| \leq \tfrac{c}{50},$$

where A, B and C are defined similarly to the $n = 3$ case. Now defining

$$r_{p,m} = \left\{(j_1, j_2) : j_1^2 + j_2^2 + (p - j_1 - j_2)^2 = m\right\}, \tag{3.20}$$

we can write the left-hand side of (3.15) as

$$\underbrace{\sum_p \sum_m \sum_{(j_1,j_2)\in r_{p,m}} \sum_\gamma \sum_{(j_4,j_5)\in r_{p,m+\gamma}} \sum_q \sum_{k_1,k_2} \sum_{k_4,k_5}}_{\text{s.t.}\left|A+2\delta^{1/2}B+\delta C\right|\leq\frac{c}{50}} a_{j_1k_1}a_{j_2k_2}a_{(p-j_1-j_2)(q-k_1-k_2)}a_{j_4k_4}a_{j_5k_5}a_{(p-j_4-j_5)(q-k_4-k_5)}.$$

3.2 The constant must depend on δ

We shall now construct examples showing that the endpoint inequalities (3.5) and (3.6) cannot hold with $A_\delta = C$.

We suppose that $a_{jk} = a_j$ is nonzero only if $k = k^*$, so there is at most one nonzero blob in each slab, and it is in the same position on each slab. Then for the $n = 3$ case, (3.5) becomes

$$\sum_{j_1,j_2,j_3,j_4} a_{j_1}a_{j_2}\overline{a_{j_3}a_{j_4}}\phi^4\left(\frac{1}{\delta}(w_{j_1k^*}+w_{j_2k^*}-w_{j_3k^*}-w_{j_4k^*})\right) \le A_\delta^4\left(\sum_j |a_j|^2\right)^2. \quad (3.21)$$

This is closely related to an example of Bourgain [Bou93, p118] which, when adapted to our context, shows that some dependence on δ (e.g. a $\delta^{-\varepsilon}$ factor) is *necessary* for both the $n=2$ and $n=3$ versions of the inequality.

We will now record the details of this argument for the $n=3$ case.

Theorem 3.8. *The inequality* (3.21) *(and hence* (3.5)*) does not hold with $A_\delta = C$.*

Proof. We consider the example with

$$a_j = \begin{cases} 1 & \text{if } j=(j_1,j_2) \text{ with } j_1,j_2\in\{1,\dots,N\} \\ 0 & \text{otherwise.}\end{cases}$$

The right-hand side of (3.21) is then $A_\delta^4 N^4$. Note that since $|j| \le \frac{1}{\beta_1}\delta^{-1/2}$, we can take $N \sim \delta^{-1/2}$.

Since all the terms are positive, a lower bound for the left-hand side of (3.21) is given by the subset of terms with $j_1+j_2=j_3+j_4$ and $|j_1|^2+|j_2|^2=|j_3|^2+|j_4|^2$, i.e.

$$\text{LHS (3.21)} \ge \sum_{\substack{j_1+j_2=j_3+j_4\\ |j_1|^2+|j_2|^2=|j_3|^2+|j_4|^2}} a_{j_1}a_{j_2}a_{j_3}a_{j_4}\underbrace{\phi^4(0)}_{\ge 1} \ge \|f\|_4^4$$

where $f:[0,1]^3\to\mathbb{R}_{\ge 0}$ is given by

$$\begin{aligned} f(x_1,x_2,x_3) &= \sum_{n_1=1}^{N}\sum_{n_2=1}^{N} e^{2\pi i(n_1x_1+n_2x_2+(n_1^2+n_2^2)x_3)} \\ &= \sum_{n_1=1}^{N} e^{2\pi i(n_1x_1+n_1^2x_3)}\sum_{n_2=1}^{N} e^{2\pi i(n_2x_2+n_2^2x_3)}, \end{aligned}$$

and the norm $\|\cdot\|_4$ is $\|\cdot\|_{L^4([0,1]^3)}$. Following the argument outlined for the $n=2$ case in [Bou93, p118], we shall show that

$$\|f\|_4^4 \gtrsim N^4\log N \quad (3.22)$$

which means (3.21) cannot hold for large N with $A_\delta = C$.

To establish (3.22), let N be fixed; we then consider integers q, a, b_1 and b_2 such that $q < N^{1/2}$ is odd, $1 \leq a < q$ with $\gcd(a, q) = 1$, and $0 \leq b_1, b_2 < q$.

Defining the rectangles

$$\mathcal{M}(q, a, b_1, b_2) = \left\{ (x_1, x_2, x_3) \; : \; \begin{array}{c} |x_1 - \frac{b_1}{q}| \lesssim \frac{1}{N}, \quad |x_2 - \frac{b_2}{q}| \lesssim \frac{1}{N}, \\ |x_3 - \frac{a}{q}| \lesssim \frac{1}{N^2} \end{array} \right\}, \tag{3.23}$$

we see that they all have size $\sim \frac{1}{N^4}$ and do not overlap as q, a, b_1, b_2 vary:

- Suppose $x \in \mathcal{M}(q, a, b_1, b_2)$ and $x \in \mathcal{M}(q', a', b'_1, b'_2)$. Note that

  $$\left| \frac{a}{q} - \frac{a'}{q'} \right| = \left| \frac{aq' - a'q}{qq'} \right| \geq \frac{1}{N} |aq' - a'q|.$$

  If $aq' \neq a'q$ then we have a contradiction as this gives $\left| \frac{a}{q} - \frac{a'}{q'} \right| \geq \frac{1}{N}$, so x_3 cannot simultaneously be within $\lesssim \frac{1}{N^2}$ of both fractions.

- If $aq' = a'q$ then we have $a = \frac{a'q}{q'}$. Now since $\gcd(a', q') = 1$, we must have $q' \mid q$, hence $q = q'q''$. But then $a = \frac{a'q'q''}{q'} = a'q''$ so $\gcd(a, q) = q''$ which must be 1, giving $a = a'$ and $q = q'$.

- It remains to see that we must have $b_1 = b'_1$ and $b_2 = b'_2$. But, for instance,

  $$\left| \frac{b_1}{q} - \frac{b'_1}{q} \right| = \frac{1}{q} |b_1 - b'_1| \geq \frac{1}{N}$$

  unless $b_1 = b'_1$.

Since these are disjoint regions,

$$\int |f|^4 \, dx \geq \sum_{\substack{3 \leq q < N^{1/2} \\ q \text{ odd}}} \; \sum_{\substack{1 \leq a < q \\ \gcd(a,q)=1}} \; \sum_{0 \leq b_1 < q} \; \sum_{0 \leq b_2 < q} \int_{\mathcal{M}(q,a,b_1,b_2)} |f|^4 \, dx. \tag{3.24}$$

The key is then to estimate $|f|$ in the region $\mathcal{M}(q, a, b_1, b_2)$, and in fact we have

$$\left| \sum_{n=1}^{N} e^{2\pi i (n x_i + n^2 x_3)} \right| \gtrsim \frac{N}{\sqrt{q}} \quad \text{for } i = 1, 2 \tag{3.25}$$

hence

$$|f| \gtrsim \frac{N^2}{q}.$$

We will obtain (3.25) from the result

$$\left| \sum_{n=1}^{q} e^{2\pi i (n\frac{b}{q} + n^2 \frac{a}{q})} \right| = q^{1/2} \tag{3.26}$$

which comes initially for q a prime power (e.g. in [Car06, Proposition 1]). This is easily extended to all odd q by multiplicativity — denoting the sum $G(a, b, q)$ we have that when $\gcd(c, d) = 1$,

$$G(a, b, cd) = G(ac, b, d)G(ad, b, c).$$

Furthermore, (3.26) holds with the sum running over any block of q consecutive integers; indeed, each such block sums to exactly the same complex number. This is because we are only concerned with the value of $an^2 + bn \mod q$.

We now split the sum in (3.25) into $\sim \frac{N}{q}$ blocks of length q. The $(j+1)$th block is then

$$\begin{aligned}
\sum_{n=jq+1}^{(j+1)q} e^{2\pi i (nx + n^2 t)} &= \sum_{m=1}^{q} e^{2\pi i \left((m+jq)x + (m+qj)^2 t\right)} \\
&= e^{2\pi i (jqx + j^2 q^2 t)} \sum_{m=1}^{q} e^{2\pi i m (x + 2jqt)} e^{2\pi i m^2 t} \\
&= e^{2\pi i (jq\beta + j^2 q^2 \alpha)} \sum_{m=1}^{q} e^{2\pi i m \frac{b}{q}} e^{2\pi i m (\beta + 2jq\alpha)} e^{2\pi i m^2 \frac{a}{q}} e^{2\pi i m^2 \alpha}
\end{aligned}$$

where we have used $x = \frac{b}{q} + \beta$, $t = \frac{a}{q} + \alpha$. Note that in light of (3.23), we have $|\beta| \lesssim \frac{1}{N}$ and $|\alpha| \lesssim \frac{1}{N^2}$.

The difference between the jth block and the known sum (3.26) is then

$$\begin{aligned}
&\sum_{n=jq+1}^{(j+1)q} e^{2\pi i (nx + n^2 t)} - \sum_{n=jq+1}^{(j+1)q} e^{2\pi i (n\frac{b}{q} + n^2 \frac{a}{q})} \\
&= e^{2\pi i (jq\beta + j^2 q^2 \alpha)} \sum_{m=1}^{q} e^{2\pi i m \frac{b}{q}} e^{2\pi i m (\beta + 2jq\alpha)} e^{2\pi i m^2 \frac{a}{q}} e^{2\pi i m^2 \alpha} - \sum_{m=1}^{q} e^{2\pi i (m\frac{b}{q} + m^2 \frac{a}{q})} \\
&= e^{2\pi i (jq\beta + j^2 q^2 \alpha)} \left(\sum_{m=1}^{q} e^{2\pi i m \frac{b}{q}} e^{2\pi i m (\beta + 2jq\alpha)} e^{2\pi i m^2 \frac{a}{q}} e^{2\pi i m^2 \alpha} - \sum_{m=1}^{q} e^{2\pi i (m\frac{b}{q} + m^2 \frac{a}{q})} \right) \\
&\quad + \left(e^{2\pi i (jq\beta + j^2 q^2 \alpha)} - 1 \right) \sum_{m=1}^{q} e^{2\pi i (m\frac{b}{q} + m^2 \frac{a}{q})} \\
&= I + II.
\end{aligned}$$

We now estimate each term, making use of the fact that $|e^{2\pi i x} - 1| \lesssim |x|$. Firstly,

$$|II| \lesssim \left(|jq\beta| + |j^2q^2\alpha|\right) q^{1/2} \lesssim q^{1/2}$$

by (3.26) and the properties of α and β. Similarly, for I we use

$$\left|\sum_m (e^{x_m} - e^{y_m})\right| \leq \sum_m |e^{x_m} - e^{y_m}| \lesssim \sum_m |x_m - y_m|$$

which gives

$$\begin{aligned}|I| &\lesssim \sum_{m=1}^{q} \left|m(\beta + 2jq\alpha) + m^2\alpha\right| \\ &\lesssim q^2|\beta + 2jq\alpha| + q^3|\alpha| \\ &\lesssim 1.\end{aligned}$$

So overall, the jth block differs from the known sum by $\leq Cq^{1/2}$. Note that the constants in (3.23) should be taken sufficiently small, so that $C < 1$.

Consider taking the sum of k blocks. This will accumulate an error of $kCq^{1/2}$ from the known sums, which together have size $kq^{1/2}$. Thus the sum of k blocks has size $\geq kq^{1/2}(1 - C)$. Returning to (3.25), which has a sum of $\sim \frac{N}{q}$ blocks, we see

$$\left|\sum_{n=1}^{N} e^{2\pi i(nx+n^2t)}\right| \geq \frac{N}{q} q^{1/2}(1 - C) \gtrsim \frac{N}{\sqrt{q}}.$$

Note that after splitting into blocks, there may be a 'remainder' with as many as $q - 1$ terms, but this does not affect the estimate since $\frac{N}{\sqrt{q}} \gtrsim q - 1$.

Now applying this in (3.24), we have

$$\begin{aligned}\int |f|^4\, dx &\gtrsim \sum_{\substack{3 \leq q < N^{1/2} \\ q \text{ odd}}} \phi(q) q^2 \frac{1}{N^4} \frac{N^8}{q^4} \\ &= N^4 \sum_{\substack{3 \leq q < N^{1/2} \\ q \text{ odd}}} \frac{\phi(q)}{q^2} \end{aligned} \tag{3.27}$$

where ϕ is Euler's totient function. It now remains to estimate this sum.

First observe that by Möbius inversion [HW08, Theorem 266],

$$\phi(n) = \sum_{d|n} d \cdot \mu\left(\frac{n}{d}\right)$$

where μ is the Möbius function; this takes values in $\{-1, 0, 1\}$. From this formula we find

$$\sum_{\substack{k=1 \\ k \text{ odd}}}^{n} \frac{\phi(k)}{k^2} = \sum_{k=1}^{n} \sum_{\substack{d'|k \\ k=dd'}} \frac{\mu(d)}{kd} = \sum_{\substack{d=1 \\ d \text{ odd}}}^{n} \frac{\mu(d)}{d^2} \sum_{\substack{d'=1 \\ d' \text{ odd}}}^{\lfloor n/d \rfloor} \frac{1}{d'}.$$

Now we have

$$\sum_{\substack{d'=1 \\ d' \text{ odd}}}^{\lfloor n/d \rfloor} \frac{1}{d'} \gtrsim \log\left\lfloor \frac{n}{d} \right\rfloor + 1 \gtrsim \log n - \log d + 1$$

so that

$$\sum_{\substack{k=1 \\ k \text{ odd}}}^{n} \frac{\phi(k)}{k^2} \gtrsim \log n \sum_{\substack{d=1 \\ d \text{ odd}}}^{n} \frac{\mu(d)}{d^2} + \sum_{\substack{d=1 \\ d \text{ odd}}}^{n} \frac{\mu(d)}{d^2}(1 - \log d).$$

Both of these sums are bounded away from zero; the first term in each is 1, and even if all the remaining terms are negative, they are so small that the sum will remain positive. Hence

$$\sum_{\substack{k=1 \\ k \text{ odd}}}^{n} \frac{\phi(k)}{k^2} \gtrsim \log n.$$

Using this in (3.27), we obtain (3.22). ■

An almost identical argument for $n = 2$ gives

Theorem 3.9. *The inequality* (3.6) *does not hold with* $A_\delta = C$.

3.3 A number-theoretic lemma

In order to proceed, we need to know more about the size of the sets $R_{p,m}$ and $r_{p,m}$ defined in (3.19) and (3.20) respectively. The following results are sketched in [Bou93, Prop 3.6 & Prop 2.36]; we shall now prove them in detail.

3.3.1 The size of $R_{p,m}$

Lemma 3.10. *Given* $\varepsilon > 0$, *there is a constant* C_ε *so that*

$$|R_{p,m}| \leq C_\varepsilon m^\varepsilon.$$

Proof. If $j \in R_{p,m}$ then $|j|^2 + |p-j|^2 = m$, which can be rewritten as

$$(2j_1 - p_1)^2 + (2j_2 - p_2)^2 = 2m - |p|^2.$$

So $X_1 = 2j_1 - p_1$ and $X_2 = 2j_2 - p_2$ are integers such that

$$\begin{aligned} X_1^2 + X_2^2 &= 2m - |p|^2 \\ \text{i.e.} \quad (X_1 + iX_2)(X_1 - iX_2) &= 2m - |p|^2. \end{aligned}$$

Thus $j \in R_{p,m}$ gives rise to divisors $X_1 \pm iX_2$ of $2m - |p|^2$ in the integral domain $\mathbb{Z}[i]$, known as the *Gaussian integers*. We see from Proposition 3.11 that the number of such divisors is

$$\lesssim (2m - |p|^2)^\varepsilon \lesssim m^\varepsilon,$$

which establishes the result. ■

Proposition 3.11. *For $A \in \mathbb{Z}$,*

$$\#\{\textit{divisors of } A\} \lesssim \exp\left(c\frac{\log A}{\log\log A}\right) \lesssim |A|^\varepsilon$$

where the divisors are in $\mathbb{Z}[i]$.

In order to prove this result, we first look more closely at the properties of $\mathbb{Z}[i]$.

We have a norm on $\mathbb{Z}[i]$ given in terms of the norm on $\mathbb{C}$,

$$N(a+bi) = |a+ib|^2 = a^2 + b^2$$

and this norm is multiplicative. So if $z \in \mathbb{Z}[i]$ divides A, we have

$$zz' = A \quad \Rightarrow \quad N(z)N(z') = N(A) \quad \Rightarrow \quad N(z)|N(A).$$

Hence to count the number of divisors of A, we can take each divisor n of $N(A) = A^2$ in turn and count all the elements of $\mathbb{Z}[i]$ with norm n; indeed this will overcount, since not all such elements are necessarily divisors.

This can be summarised as

$$\{\text{divisors of } A\} \subseteq \bigcup_{n|N(A)} \{z \in \mathbb{Z}[i] \,:\, N(z) = n\}. \tag{3.28}$$

Now the union has $d(N(A)) = d(A^2)$ terms, where d is the familiar "number of divisors" function on $\mathbb{N}$. The behaviour of this is well-known:

Proposition 3.12. *Given $\varepsilon > 0$, $\exists n_0(\varepsilon)$ s.t. for all $n > n_0(\varepsilon)$,*

$$d(n) \leq \exp\left((1+\varepsilon)\log 2 \frac{\log n}{\log\log n}\right).$$

Proof. See [HW08, p345]. ∎

So in (3.28) the number of terms in the union is $\lesssim \exp\left(c\frac{\log A}{\log\log A}\right)$. It remains to estimate the size of each set in the union; this is again done using $d(n)$.

Proposition 3.13.

$$\#\{z \in \mathbb{Z}[i] \,:\, N(z) = n\} \lesssim d(n).$$

Proof. We are counting $z = a + ib$ so that $N(z) = a^2 + b^2 = n$, hence

$$n = (a+ib)\overline{(a+ib)} = (a+ib)(a-ib). \tag{3.29}$$

Note that we can also factorise $n \in \mathbb{N}$ as

$$n = 2^\alpha \prod_{p \equiv 1 \bmod 4} p^r \prod_{q \equiv 3 \bmod 4} q^s$$

where the $p, q \in \mathbb{N}$ are prime, and the r, s vary in the products. Breaking down these terms into primes of $\mathbb{Z}[i]$ (see [HW08, Theorem 252]) we obtain

$$n = ((1+i)(1-i))^\alpha \prod_{p \equiv 1 \bmod 4} \left((u_p + iv_p)(u_p - iv_p)\right)^r \prod_{q \equiv 3 \bmod 4} q^s.$$

Now from (3.29) we must decide how to split these powers of primes between the factors $a \pm ib$. If we have

$$a + ib = i^t (1+i)^{\alpha_1} (1-i)^{\alpha_2} \prod_{p \equiv 1 \bmod 4} (u_p + iv_p)^{r_1} (u_p - iv_p)^{r_2} \prod_{q \equiv 3 \bmod 4} q^{s_1}$$

then conjugating throughout gives

$$a - ib = i^{-t} (1-i)^{\alpha_1} (1+i)^{\alpha_2} \prod_{p \equiv 1 \bmod 4} (u_p - iv_p)^{r_1} (u_p + iv_p)^{r_2} \prod_{q \equiv 3 \bmod 4} q^{s_1}.$$

Note that there is no choice in the splitting of the qs; these must be split evenly ($s_1 + s_1 = s$) and this is possible since each exponent s must be even if n can be expressed as a sum of squares. We also see that changing α_1, α_2 just produces associates, so we can ignore these choices and multiply the remaining number of choices by 4 (for the units $\pm 1, \pm i$).

Lastly, there are $r+1$ choices of r_1, r_2 so that $r_1 + r_2 = r$, and we make these for each $p \equiv 1 \bmod 4$ in the prime factorisation. Overall this gives

$$4 \prod_{p \equiv 1 \bmod 4} (r+1) = 4d\left(\prod_{p \equiv 1 \bmod 4} p^r\right) \leq 4d(n)$$

choices of $z = a + ib$ with $N(z) = n$. ∎

Proof of Proposition 3.11. We have

$$\{\text{divisors of } A\} \subseteq \bigcup_{n \mid N(A)} \{z \in \mathbb{Z}[\omega] \,:\, N(z) = n\}$$

hence

$$\#\{\text{divisors of } A\} \leq d(A^2) \times \max_{n \mid N(A)} d(n) \lesssim \exp\left(c \frac{\log A}{\log \log A}\right).$$

It remains to see that this log term can be replaced with A^{ε}. Now

$$g(n) \lesssim \exp\left(c \frac{\log n}{\log \log n}\right) \Rightarrow \frac{\log g(n)}{\log n} \lesssim \frac{c}{\log \log n} \to 0 \text{ as } n \to \infty.$$

So given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\frac{\log g(n)}{\log n} < \varepsilon$, $\forall n \geq N$.

But $g(n) = n^{\frac{\log g(n)}{\log n}}$, so we have $g(n) < n^{\varepsilon}$ for $n \geq N$, i.e. $g(n) = O(n^{\varepsilon})$. ∎

3.3.2 The size of $r_{p,m}$

Lemma 3.14. *Given $\varepsilon > 0$, there is a constant C_{ε} so that $|r_{p,m}| \leq C_{\varepsilon} m^{\varepsilon}$.*

Proof. If $(j_1, j_2) \in r_{p,m}$ then $j_1^2 + j_2^2 + (p - j_1 - j_2)^2 = m$, which can be rewritten as

$$\big(3(j_1 + j_2) - 2p\big)^2 + 3(j_1 - j_2)^2 = 6m - 2p^2.$$

So $X_1 = 3(j_1 + j_2) - 2p$ and $X_2 = j_1 - j_2$ are integers such that

$$X_1^2 + 3X_2^2 = 6m - 2p^2$$
$$\text{i.e.} \quad (X_1 + \omega X_2)(X_1 - \omega X_2) = 6m - 2p^2,$$

where $\omega = e^{\frac{2\pi i}{3}}$. Thus $j \in r_{p,m}$ gives rise to divisors $X_1 \pm \omega X_2$ of $6m - 2p^2$ in the *Eisenstein integers*, $\mathbb{Z}[\omega]$ (see [IR82, Chapter 9, §1]). Here we use the norm

$$N(a + \omega b) = (a + \omega b)(a - \omega b) = a^2 - ab + b^2$$

and the rest of the argument runs as before, except with Proposition 3.15 used in place of Proposition 3.13. ∎

Proposition 3.15.

$$\#\{z \in \mathbb{Z}[\omega] \, : \, N(z) = n\} \lesssim d(n).$$

Proof. We use the fact that

$$n = a^2 - ab + b^2 = (a + b\omega)\overline{(a + b\omega)}.$$

The idea is now to take the (unique) decomposition of n into primes of $\mathbb{Z}[\omega]$, then count the possible ways to split these up between the two factors $(a + b\omega)$ and $\overline{(a + b\omega)}$. We note the following classification [IR82, Prop 9.1.4] of the primes of $\mathbb{Z}[\omega]$ in terms of primes $p \in \mathbb{N}$.

- $3 = (-1 - 2\omega)(1 + 2\omega)$ and these two factors are primes in $\mathbb{Z}[\omega]$.
- If $p \equiv 2 \bmod 3$ then $p \in \mathbb{Z}[\omega]$ is prime.
- If $p \equiv 1 \bmod 3$ then $p = q\overline{q}$ with $q \in \mathbb{Z}[\omega]$ prime.

Using this, we can write the prime decomposition of n in $\mathbb{Z}[\omega]$ as

$$n = ((-1 - 2\omega)(1 + 2\omega))^{\alpha} \times \prod_{p \equiv 2 \bmod 3} p^r \times \prod_{p \equiv 1 \bmod 3} (q\overline{q})^s$$

where the $p \in \mathbb{N}$ are prime, and the r and s vary in the products.

Now choosing s_1, s_2 so that $s_1 + s_2 = s$ and similarly for α_1, α_2, we put

$$a + b\omega = (-1 - 2\omega)^{\alpha_1}(1 + 2\omega)^{\alpha_2} \times \prod_{p \equiv 2 \bmod 3} p^r \times \prod_{p \equiv 1 \bmod 3} q^{s_1}\overline{q}^{s_2}$$

and consequently have

$$\overline{a+b\omega} = (1+2\omega)^{\alpha_1}(-1-2\omega)^{\alpha_2} \times \prod_{p\equiv 2 \bmod 3} p^r \times \prod_{p \equiv 1 \bmod 3} \overline{q}^{s_1} q^{s_2}.$$

The problem now reduces to counting the choices of s_1, s_2 and α_1, α_2, since these determine the a and b. Note that varying α_1 and α_2 will only produce associates of a certain $a+b\omega$, so we can ignore this contribution by taking $6\times$ the remaining contribution (since there are 6 units). What remains are the $s+1$ possible choices of s_1, s_2; overall this gives

$$6 \prod_{p\equiv 1 \bmod 3} (s+1) = 6d\left(\prod_{p\equiv 1 \bmod 3} p^s\right) \leq 6d(n)$$

different $a+b\omega$ with norm n. ∎

3.4 Test case: one blob per slab

3.4.1 Same position on each slab

We suppose that $a_{jk} = a_j$ is nonzero only if $k = k^*$, so there is at most one nonzero blob in each slab, and it is in the same position on each slab. Then for the $n = 3$ case, (3.5) becomes

$$\sum_{j_1,j_2,j_3,j_4} a_{j_1}a_{j_2}\overline{a_{j_3}a_{j_4}}\phi^4\left(\frac{1}{\delta}(w_{j_1k^*} + w_{j_2k^*} - w_{j_3k^*} - w_{j_4k^*})\right) \leq A_\delta^4\left(\sum_j |a_j|^2\right)^2. \quad (3.30)$$

Now we can express the argument of ϕ^4 (as in (3.16)) as

$$\frac{1}{\delta}\left(\beta_1\delta^{1/2}(j_1+j_2-j_3-j_4),\ \begin{array}{c}\beta_1\delta^{1/2}\alpha\cdot(j_1+j_2-j_3-j_4)\\ +\frac{1}{2}\beta_1^2\delta(|j_1|^2+|j_2|^2-|j_3|^2-|j_4|^2)\\ +\beta_1\beta_2\delta^{3/2}k^*\cdot(j_1+j_2-j_3-j_4)\end{array}\right)$$

Since we can suppose ϕ is radial, we can write $\phi^4(x) = \psi(|x|)$ for some $\psi \in \mathcal{S}(\mathbb{R})$. Thus

$$\text{LHS (3.30)} = \sum_{j_1,j_2,j_3,j_4} a_{j_1}a_{j_2}\overline{a_{j_3}a_{j_4}}\psi\left(\sqrt{\beta_1^2\delta^{-1}|j_1+j_2-j_3-j_4|^2 + \text{other terms}}\right)$$

We now use the triangle inequality and break the sum into two parts, according to whether or not $j_1 + j_2 = j_3 + j_4$.

- When $j_1 + j_2 \neq j_3 + j_4$, the argument of ψ is always larger than $\delta^{-1/2}$ (since the "other terms" are certainly positive), so

$$\begin{aligned}\left|\sum_{j_1+j_2\neq j_3+j_4} a_{j_1}a_{j_2}\overline{a_{j_3}a_{j_4}}\psi(\dots)\right| &\leq \psi(\delta^{-1/2})\sum_{j_1+j_2\neq j_3+j_4}|a_{j_1}a_{j_2}\overline{a_{j_3}a_{j_4}}|\\ &= \psi(\delta^{-1/2})\sum_{j_1}|a_{j_1}|\sum_{j_2}|a_{j_2}|\sum_{j_3}|a_{j_3}|\sum_{j_4\neq j_1+j_2-j_3}|a_{j_4}|\\ &\leq \psi(\delta^{-1/2})\left(\sum_j |a_j|\right)^4.\end{aligned}$$

  Now, $\sum_j |a_j| \leq (\#j's)\max|a_j| \sim \delta^{-1}\max|a_j|$, so the above is bounded by

$$\psi(\delta^{-1/2})(\delta^{-1})^4(\max|a_j|)^4.$$

  Note that as $\psi \in \mathcal{S}(\mathbb{R})$ we have $\psi(\delta^{-1/2})(\delta^{-1})^4 \leq C$, and since

$$\left(\sum_j |a_j|^2\right)^2 = \Big((\max|a_j|)^2 + \text{other terms}\Big)^2 = (\max|a_j|)^4 + \text{other terms}$$

  we get the desired bound for this part.

- When $j_1 + j_2 = j_3 + j_4$, we have

$$\begin{aligned}&\sum_{j_1+j_2=j_3+j_4} a_{j_1}a_{j_2}\overline{a_{j_3}a_{j_4}}\psi\left(\tfrac{1}{2}\beta_1^2\Big(|j_1|^2+|j_2|^2-|j_3|^2-|j_4|^2\Big)\right)\\ &\qquad = \sum_\gamma\sum_p\sum_m\sum_{j_1\in R_{p,m}}\sum_{j_3\in R_{p,m+\gamma}} a_{j_1}a_{p-j_1}\overline{a_{j_3}a_{p-j_3}}\psi(\tfrac{1}{2}\beta_1^2\gamma),\end{aligned} \tag{3.31}$$

  using the same notation as in (3.18). Now for each *fixed* γ,

$$\begin{aligned}&\sum_p\sum_m\sum_{j_1\in R_{p,m}} a_{j_1}a_{p-j_1}\sum_{j_3\in R_{p,m+\gamma}}\overline{a_{j_3}a_{p-j_3}}\\ &\overset{\text{C-S}}{\leq} \sum_p\sum_m |R_{p,m}|^{1/2}\left(\sum_{j_1\in R_{p,m}}|a_{j_1}a_{p-j_1}|^2\right)^{1/2}|R_{p,m+\gamma}|^{1/2}\left(\sum_{j_3\in R_{p,m+\gamma}}|a_{j_3}a_{p-j_3}|^2\right)^{1/2}\\ &\overset{\text{C-S}}{\lesssim} \delta^{-\varepsilon}\left(\sum_p\sum_m\sum_{j_1\in R_{p,m}}|a_{j_1}a_{p-j_1}|^2\right)^{1/2}\left(\sum_p\sum_m\sum_{j_3\in R_{p,m+\gamma}}|a_{j_3}a_{p-j_3}|^2\right)^{1/2},\end{aligned} \tag{3.32}$$

since, by Lemma 3.10, $|R_{p,m}| \lesssim m^{\varepsilon}$ for any $\varepsilon > 0$, and we have $m \lesssim \delta^{-1}$. Now if we reorder the sums, e.g. $\sum_p \sum_{j_1} \sum_m$, we see that m is already determined so we end up with

$$\delta^{-\varepsilon} \left(\sum_p \sum_{j_1} |a_{j_1} a_{p-j_1}|^2 \right)^{1/2} \left(\sum_p \sum_{j_3} |a_{j_3} a_{p-j_3}|^2 \right)^{1/2} = \delta^{-\varepsilon} \left(\sum_j |a_j|^2 \right)^2 .$$

Using this on (3.31), we find

$$\sum_{j_1+j_2=j_3+j_4} \cdots \lesssim \delta^{-\varepsilon} \left(\sum_j |a_j|^2 \right)^2 \sum_\gamma \psi(\tfrac{1}{2}\beta_1^2 \gamma)$$

but again using $\psi \in \mathcal{S}(\mathbb{R})$, we know $\psi(\frac{1}{2}\beta_1^2\gamma) \lesssim \frac{1}{|\gamma|^2}$, so the above sum in γ converges.

Putting these two parts together, we obtain (3.30) with $A_\delta = C_\varepsilon \delta^{-\varepsilon}$. From this, we have

Theorem 3.16 ($n = 3$ Fixed Finite Blobs). *When a_{jk} is zero for all but a fixed finite set of k, the inequality* (3.5) *is true for all $\varepsilon > 0$ with $A_{\delta,c} = C_\varepsilon \delta^{-\varepsilon}$.*

Proof. Putting each k which has nonzero a_{jk} into its own set K_i, we appeal to the argument used in the proof of Proposition 3.4 — since for each $i \in \Lambda$ the "separated" inequality is of the form (3.30), and $|\Lambda| = \#\{k\text{s}\}$ is a constant independent of δ, that argument gives (3.5) with $A_{\delta,c} = C_\varepsilon \delta^{-\varepsilon}$. ∎

Remark 3.17. Note that the above argument works with $a_{jk} \in \mathbb{C}$; we did not need to assume at any point that $a_{jk} \geq 0$.

Observe, however, that the right-hand side of (3.30) is all in terms of $|a_{jk}|$, so the worst case is in fact when $a_{jk} \geq 0$ as this precludes any cancellation on the left-hand side. ♢

For the $n = 2$ case, the inequality is

$$\sum_{j_1,j_2,j_3,j_4,j_5,j_6} a_{j_1} a_{j_2} a_{j_3} \overline{a_{j_4} a_{j_5} a_{j_6}} \phi^6 \left(\frac{1}{\delta} (w_{j_1 k^*} + w_{j_2 k^*} + w_{j_3 k^*} - w_{j_4 k^*} - w_{j_5 k^*} - w_{j_6 k^*}) \right)$$
$$\leq A_\delta^6 \left(\sum_j |a_j|^2 \right)^3 .$$

The argument given above is easily adapted to this case (for instance, using sets $r_{p,m}$ in place of $R_{p,m}$ and making use of Lemma 3.14), giving

Theorem 3.18 ($n = 2$ Fixed Finite Blobs). *When a_{jk} is zero for all but a fixed finite set of k, the inequality* (3.6) *is true for all $\varepsilon > 0$ with $A_{\delta,c} = C_\varepsilon \delta^{-\varepsilon}$.*

3.4.2 Arbitrary position on each slab

With $a_{jk} = a_j$ nonzero only for one $k = k_j$, the argument of ϕ^4 is not as easy to handle as in the "same position" case.

Indeed, the question becomes (in the $n = 3$ case)

$$\sum_{j_1,j_2,j_3,j_4} a_{j_1} a_{j_2} \overline{a_{j_3} a_{j_4}} \phi^4 \left(\frac{1}{\delta} (w_{j_1 k_{j_1}} + w_{j_2 k_{j_2}} - w_{j_3 k_{j_3}} - w_{j_4 k_{j_4}}) \right) \le A_\delta^4 \left(\sum_j |a_j|^2 \right)^2,$$

and if we repeat the argument used on (3.30) we can write the left-hand side of this as

$$\sum_{j_1,j_2,j_3,j_4} a_{j_1} a_{j_2} \overline{a_{j_3} a_{j_4}} \psi \left(\left| \left(\begin{array}{c} \delta^{-1/2}(j_1 + j_2 - j_3 - j_4) \\ +(k_{j_1} + k_{j_2} - k_{j_3} - k_{j_4}) \end{array}, \begin{array}{c} \delta^{-1/2}\alpha \cdot (j_1 + j_2 - j_3 - j_4) \\ +\frac{1}{2}(|j_1|^2 + |j_2|^2 - |j_3|^2 - |j_4|^2) \\ +\delta^{1/2}(j_1 \cdot k_{j_1} + j_2 \cdot k_{j_2} - j_3 \cdot k_{j_3} - j_4 \cdot k_{j_4}) \\ +\frac{1}{2}\delta(|k_{j_1}|^2 + |k_{j_2}|^2 - |k_{j_3}|^2 - |k_{j_4}|^2) \end{array} \right) \right|^2 \right).$$

As before, the terms with $j_1 + j_2 \neq j_3 + j_4$ are easily dealt with. This is because the argument of ψ is at least $|\delta^{-1/2}(j_1 + j_2 - j_3 - j_4) + (k_{j_1} + k_{j_2} - k_{j_3} - k_{j_4})|^2$, and if we use slab separation we can ensure that the j_i term dominates, so this is always $\gtrsim \delta^{-1}$.

Now the main term, with $j_1 + j_2 = j_3 + j_4$, can be rewritten as

$$\sum_\gamma \sum_p \sum_m \sum_{j_1 \in R_{p,m}} \sum_{j_3 \in R_{p,m+\gamma}} a_{j_1} a_{p-j_1} \overline{a_{j_3} a_{p-j_3}} \psi(X_{\gamma,p,j_1,j_3}), \tag{3.33}$$

where

$$\begin{aligned} X_{\gamma,p,j_1,j_3} &= |k_{j_1} + \cdots - k_{j_4}|^2 \\ &\quad + \frac{1}{4} \left| \gamma + 2\delta^{1/2}(j_1 \cdot k_{j_1} + \cdots - j_4 \cdot k_{j_4}) + \delta(|k_{j_1}|^2 + \cdots - |k_{j_4}|^2) \right|^2. \end{aligned}$$

This is where the difficulty arises — unlike in (3.31), the argument of ψ depends on p, j_1, and j_3 (as well as the specific choice of k_j), so the Cauchy-Schwarz argument cannot be used. However, we can apply Schur's Inequality:

Lemma 3.19 (Schur's Inequality). *Given the numbers c_{jk}, x_j and y_k for $1 \le j,k \le m$, we* *have*

$$\left|\sum_{j=1}^{m}\sum_{k=1}^{n} c_{jk}x_j y_k\right| \leq \sqrt{RC}\left(\sum_{j=1}^{m}|x_j|^2\right)^{1/2}\left(\sum_{k=1}^{n}|y_k|^2\right)^{1/2}$$

where

$$R = \max_j \sum_{k=1}^{n}|c_{jk}| \quad \textit{and} \quad C = \max_k \sum_{j=1}^{m}|c_{jk}|.$$

Proof. This is Exercise 1.10 in [Ste04]. The result is obtained by applying Cauchy-Schwarz to

$$\left|\sum_{j,k} c_{jk}x_j y_k\right| \leq \sum_{j,k}|c_{jk}|^{1/2}|x_j||c_{jk}|^{1/2}|y_k|$$

which gives

$$\begin{aligned}
\text{LHS} &\leq \left(\sum_{j,k}|c_{jk}||x_j|^2\right)^{1/2}\left(\sum_{j,k}|c_{jk}||y_k|^2\right)^{1/2} \\
&= \left(\sum_{j=1}^{m}\underbrace{\left(\sum_{k=1}^{n}|c_{jk}|\right)}_{\leq R}|x_j|^2\right)^{1/2}\left(\sum_{k=1}^{n}\underbrace{\left(\sum_{j=1}^{m}|c_{jk}|\right)}_{\leq C}|y_k|^2\right)^{1/2}.
\end{aligned}$$

■

Applying this to (3.33), we obtain

$$\sum_{\gamma}\sum_{p}\sum_{m}\sqrt{RC}\left(\sum_{j_1\in R_{p,m}}|a_{j_1}a_{p-j_1}|^2\right)^{1/2}\left(\sum_{j_3\in R_{p,m+\gamma}}|a_{j_3}a_{p-j_3}|^2\right)^{1/2}$$

where

$$R = \max_{j_1\in R_{p,m}}\sum_{j_3\in R_{p,m+\gamma}}|\psi(X_{\gamma,p,j_1,j_3})|, \quad C = \max_{j_3\in R_{p,m+\gamma}}\sum_{j_1\in R_{p,m}}|\psi(X_{\gamma,p,j_1,j_3})|.$$

Pulling $\sqrt{RC}$ out of the sums in p and m gives

$$\begin{aligned}
\text{LHS (3.33)} &\leq \sum_{\gamma}\max_{p,m}\sqrt{RC}\sum_{p}\sum_{m}\left(\sum_{j_1\in R_{p,m}}|a_{j_1}a_{p-j_1}|^2\right)^{1/2}\left(\sum_{j_3\in R_{p,m+\gamma}}|a_{j_3}a_{p-j_3}|^2\right)^{1/2} \\
&\leq \delta^{-\varepsilon}\sum_{\gamma}\max_{p,m}\sqrt{RC}\left(\sum_{j}|a_j|^2\right)^2
\end{aligned}$$

after the usual Cauchy-Schwarz and rearrangement argument. Thus we are left with

Question 3.20.

$$\sum_{\gamma} \max_{p,m} \left(\max_{j_1 \in R_{p,m}} \sum_{j_3 \in R_{p,m+\gamma}} |\psi(X_{\gamma,p,j_1,j_3})| \max_{j_3 \in R_{p,m+\gamma}} \sum_{j_1 \in R_{p,m}} |\psi(X_{\gamma,p,j_1,j_3})| \right)^{1/2} \lesssim \delta^{-\varepsilon}.$$

Remark 3.21. When all $k_j = k^*$, we have $X_{\gamma,p,j_1,j_3} = \frac{1}{4}\gamma^2$, so the left-hand side is

$$\sum_{\gamma} |\psi(\tfrac{1}{4}\gamma^2)| \max_{p,m} \left(|R_{p,m+\gamma}||R_{p,m}|\right)^{1/2} \lesssim \delta^{-\varepsilon} \sum_{\gamma} |\psi(\tfrac{1}{4}\gamma^2)| \lesssim \delta^{-\varepsilon}.$$

The problem for arbitrary k_j lies in finding a good way to bound X_{γ,p,j_1,j_3}. $\Diamond$

Positive coefficients case

Even if we assume $a_{jk} \geq 0$, the conditions on the left-hand side given by (3.17) do not simplify neatly. Indeed, we get

$$\sum_{p} \sum_{m} \sum_{j_1 \in R_{p,m}} \sum_{j_3} a_{j_1} a_{p-j_1} \overline{a_{j_3} a_{p-j_3}} \leq A_{\delta,c}^4 \left(\sum_{j} |a_j|^2 \right)^2 \tag{3.34}$$

where the sum in j_3 is not simply over $R_{p,m}$ — the difficulty lies in identifying the correct set of j_3s to sum over.

In fact, putting $j_2 = p - j_1$, $j_4 = p - j_3$ and $k_{j_4} = k_{j_1} + k_{p-j_1} - k_{j_3}$ we see that j_3 must satisfy

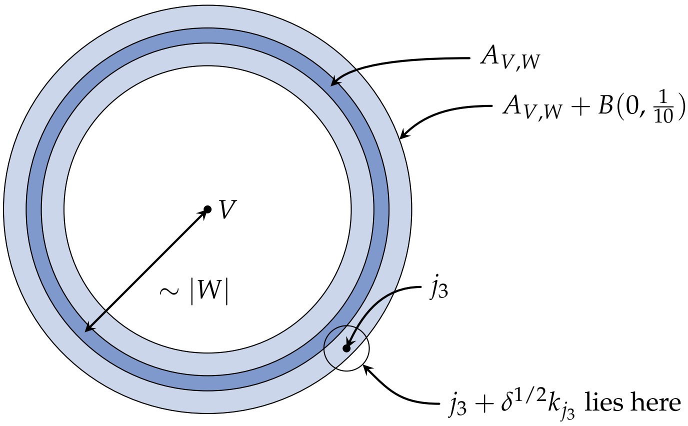
$$\begin{aligned} \left| A + 2\delta^{1/2} B + \delta C \right| \leq \tfrac{c}{50} \quad \text{with} \quad A &= 2(|j_1|^2 - |j_3|^2) + 2p \cdot (j_3 - j_1), \\ B &= j_1 \cdot (k_{j_1} - k_{p-j_1}) + p \cdot (k_{j_3} - k_{j_1}) \\ &\quad + j_3 \cdot (k_{j_1} + k_{p-j_1} - 2k_{j_3}), \\ C &= |k_{j_1}|^2 + |k_{p-j_1}|^2 - |k_{j_3}|^2 \\ &\quad - |k_{j_1} + k_{p-j_1} - k_{j_3}|^2. \end{aligned} \tag{3.35}$$

The condition (3.35) can be manipulated into the form

$$\left| \Big| \underbrace{\tfrac{p}{2} + \delta^{1/2} \tfrac{k_{j_1} + k_{p-j_1}}{2}}_{V} - (j_3 + \delta^{1/2} k_{j_3}) \Big|^2 - \Big| \underbrace{j_1 - \tfrac{p}{2} + \delta^{1/2} \tfrac{k_{j_1} - k_{p-j_1}}{2}}_{W} \Big|^2 \right| \leq \tfrac{c}{100}$$

which shows that once p and j_1 are fixed (thus determining k_{j_1} and k_{p-j_1}, and hence V and W), j_3 must be chosen so that $j_3 + \delta^{1/2} k_{j_3}$ lies in the annulus $A_{V,W}$, centred at V, with radii between $\sqrt{|W|^2 - \frac{c}{100}}$ and $\sqrt{|W|^2 + \frac{c}{100}}$.

Now consider a certain choice of p and j_1, so that $A_{V,W}$ is fixed. Since we choose k_{j_3} at the outset satisfying $|k_{j_3}| \le \frac{1}{10}\delta^{-1/2}$, we may find suitable j_3 anywhere in $A_{V,W} + B(0, \frac{1}{10})$. This is because an appropriate choice of k_{j_3} would then put $j_3 + \delta^{1/2}k_{j_3}$ in $A_{V,W}$.



Since the thickened annulus $A_{V,W} + B(0, \frac{1}{10})$ has area $\sim |W|$, it can contain $\lesssim \delta^{-1/2}$ points of $\mathbb{Z}^2$, and there will be examples where this upper bound is attained. This shows that for certain choices of the arbitrary blob locations k_j, the sum in j_3 appearing in (3.34) may have $> \delta^{\varepsilon}$ terms. This is in contrast to the situation in (3.31), and means that the Cauchy-Schwarz-based argument used to deal with that case will not work.

However, the process of fixing the values of k_{j_3} to produce these large j_3 sets can only be carried out a certain number of times before all the choices of k_j are decided. Intuitively, it seems that while certain choices of k_j will give "bad" terms in (3.34), there cannot be too many of these bad terms. This leads us to expect that the worst case is the one already considered, and experiments carried out in Maple support this — see Appendix A.3.

Question 3.22 (One blob per slab — arbitrary vs same position)**.** *Is it the case that locating the single blob on each slab in the same location gives the largest possible left-hand side? That is, do we have*

$$\sum_p \sum_m \sum_{j_1 \in R_{p,m}} \sum_{\substack{j_3 \\ s.t. j_3 + \delta^{1/2}k_{j_3} \in A_{V,W}}} a_{j_1} a_{p-j_1} \overline{a_{j_3} a_{p-j_3}} \le \sum_p \sum_m \sum_{j_1, j_3 \in R_{p,m}} a_{j_1} a_{p-j_1} \overline{a_{j_3} a_{p-j_3}}$$

for any choice of k_j?

Looking back at where the sums in question came from, namely in (3.3), we see that the question would be answered if we could show

$$\sum_{j_1,j_2,j_3,j_4} a_{j_1}a_{j_2}\overline{a_{j_3}a_{j_4}}\phi^4\left(\frac{w_{j_1k(j_1)}+w_{j_2k(j_2)}-w_{j_3k(j_3)}-w_{j_4k(j_4)}}{\delta}\right)$$
$$\leq \sum_{j_1,j_2,j_3,j_4} a_{j_1}a_{j_2}\overline{a_{j_3}a_{j_4}}\phi^4\left(\frac{w_{j_1k^*}+w_{j_2k^*}-w_{j_3k^*}-w_{j_4k^*}}{\delta}\right)$$

where $\phi \in \mathcal{S}(\mathbb{R})$ is such that $\phi \leq \chi_{B(0,2)}$ and $\phi = 1$ on $B(0,1)$, and on the left-hand side, $k : \mathbb{Z}^2 \to \mathbb{Z}^2$ is the function which selects the blob location on each slab. Now if we consider extending k to be a smooth function $\mathbb{R}^2 \to \mathbb{R}^2$, we see that

$$S[k] = \sum_{j_1,j_2,j_3,j_4} a_{j_1}a_{j_2}\overline{a_{j_3}a_{j_4}}\phi^4\left(\frac{w_{j_1k(j_1)}+w_{j_2k(j_2)}-w_{j_3k(j_3)}-w_{j_4k(j_4)}}{\delta}\right)$$

is a continuous functional in k, and Question 3.22 amounts to showing that S is maximised by constant functions.

3.5 Test case: constant on slabs

Putting $a_{jk} = \lambda_j$ for all k into (3.5), we see that the right-hand side is

$$A_\delta^4\left(\sum_j\left(|\lambda_j|^4\sum_{k_1,k_2,k_3,k_4}\phi^4\left(\frac{w_{jk_1}+w_{jk_2}-w_{jk_3}-w_{jk_4}}{\delta}\right)\right)^{1/2}\right)^2$$

so, in particular, it does not depend on the sign of the λ_j. Thus the worst case to consider is that of all $\lambda_j \geq 0$, since this will preclude any cancellation on the left-hand side, so we consider this example in (3.14).

Theorem 3.23. *When $a_{jk} = \lambda_j$ for all k, the inequality* (3.14) *(and hence* (3.5)*) is true for all $\varepsilon > 0$ with $A_{\delta,c} = C_\varepsilon\delta^{-\varepsilon}$.*

Proof. The right-hand side of (3.14) is

$$A_{\delta,c}^4\left(\sum_j\left(\sum_{k_1+k_2=k_3+k_4}|\lambda_j|^4\right)^{1/2}\right)^2 \gtrsim A_{\delta,c}^4\delta^{-3}\left(\sum_j|\lambda_j|^2\right)^2$$

since there are $\sim \delta^{-1}$ choices of each of k_1, k_2, k_3, from which k_4 is determined.

Thus from the alternative form (3.18) of the left-hand side, the question requires us to show

$$\underbrace{\sum_p \sum_m \sum_{j_1 \in R_{p,m}} \sum_\gamma \sum_{j_3 \in R_{p,m+\gamma}} \sum_q \sum_{k_1} \sum_{k_3}}_{\text{s.t.} \left|A+2\delta^{1/2}B+\delta C\right| \leq \frac{c}{50}} \lambda_{j_1} \lambda_{p-j_1} \lambda_{j_3} \lambda_{p-j_3} \lesssim A_{\delta,c}^4 \delta^{-3} \left(\sum_j |\lambda_j|^2 \right)^2$$

Now we can write the left-hand side as

$$\sum_p \sum_m \sum_{j_1 \in R_{p,m}} \sum_\gamma \sum_{j_3 \in R_{p,m+\gamma}} \# \left\{ (q, k_1, k_3) \ : \ \left| A + 2\delta^{1/2} B + \delta C \right| \leq \tfrac{c}{50} \right\} \lambda_{j_1} \lambda_{p-j_1} \lambda_{j_3} \lambda_{p-j_3}. \tag{3.36}$$

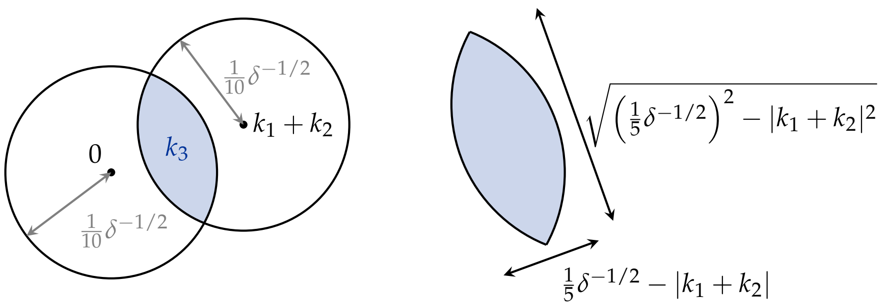
In order to count the number of (q, k_1, k_3) tuples, first suppose k_3 is the only variable which is not fixed. In that case, the condition $\left|A + 2\delta^{1/2} B + \delta C\right| \leq \frac{c}{50}$ can be viewed as defining the set of possible k_3. Since this can be rewritten in the form

$$\left| |Z|^2 - |Z + Y - k_3|^2 \right| \leq \frac{c}{100} \delta^{-1} \tag{3.37}$$

where $Z = \frac{k_2 - k_1}{2} + \delta^{-1/2} \frac{j_2 - j_1}{2}$ and $Y = k_1 + \delta^{-1/2}(j_1 - j_3)$, we see that k_3 must be chosen from the annulus with

- centre $Z + Y = \frac{1}{2} q + \frac{1}{2} \delta^{-1/2}(j_4 - j_3)$, and
- radii in the range $\left[\sqrt{|Z|^2 - \frac{c}{100}\delta^{-1}}, \sqrt{|Z|^2 + \frac{c}{100}\delta^{-1}} \right]$.

Also, from the size restrictions $|k_3| \leq \frac{1}{10}\delta^{-1/2}$ and $|k_4| = |k_3 - (k_1 + k_2)| \leq \frac{1}{10}\delta^{-1/2}$ we see that k_3 must lie in a lens-shaped region, with dimensions given by expressions involving $|k_1 + k_2|$.



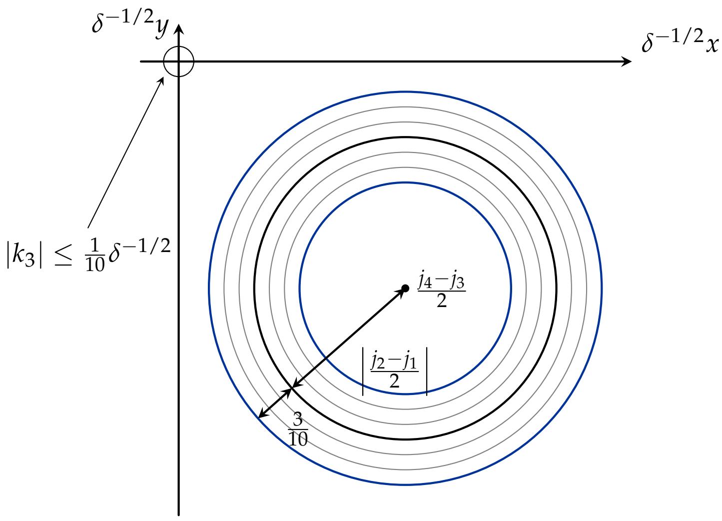
Note that we only get a contribution if the k_3 annulus overlaps the k_3 lens. We shall suppose that q and k_1 are not yet fixed, in order to determine the largest range of γ

for which this overlap could occur.

Looking back at the description of the k_3 annulus, we see that with q and k_1 free, there is some freedom in:

1. the position of the centre, which relies on $\frac{1}{2}q$,
2. the "central" radius of the annulus, coming from $|Z|$, which relies on $\frac{1}{2}(k_2 - k_1) = \frac{1}{2}(q - 2k_1)$, and
3. the amount of fattening of the annulus, since the range of radii also depends on $|Z|$.

However, in each case the amount of freedom is $\leq \frac{1}{10}\delta^{-1/2}$, so the k_3 annulus is contained in a larger annulus; namely the circle with centre $\frac{1}{2}\delta^{-1/2}(j_4 - j_3)$ and radius $\frac{1}{2}\delta^{-1/2}|j_2 - j_1|$ fattened by $\frac{3}{5}\delta^{-1/2}$.



There can only be a contribution if k_3 lies in both this large annulus and the disc of radius $\frac{1}{10}\delta^{-1/2}$ about the origin; thus the two regions must overlap, and we see that this can happen only when

$$|j_4 - j_3| \in |j_2 - j_1| + \left[-\tfrac{4}{5}, \tfrac{4}{5}\right]. \tag{3.38}$$

Let us first deal with those contributions with $|j_2 - j_1| \leq 2$. From $j_1 \in R_{p,m}$ we find

$$\begin{aligned} |j_1|^2 + |j_2|^2 &= m \\ \text{so} \quad \tfrac{1}{2}\left(|p|^2 + |j_1 - j_2|^2\right) &= m, \end{aligned}$$

so once p is chosen we have $m \in \left[\frac{1}{2}|p|^2, \frac{1}{2}|p|^2+2\right]$. Now from (3.38) we have $|j_3 - j_4| < 3$, so

$$\begin{aligned}|j_3|^2 + |j_4|^2 &= m + \gamma \\ \text{i.e.} \quad \gamma &= \tfrac{1}{2}\left(|j_3 - j_4|^2 - |j_2 - j_1|^2\right),\end{aligned}$$

which shows $-2 \le \gamma < \frac{9}{2}$. Using the crude bound of δ^{-3} for the number of (q, k_1, k_3) tuples, we can bound (3.36) by

$$\delta^{-3} \sum_p \sum_{m=\frac{1}{2}|p|^2}^{\frac{1}{2}|p|^2+2} \sum_{\gamma=-2}^{4} \sum_{j_1 \in R_{p,m}} \lambda_{j_1}\lambda_{p-j_1} \sum_{j_3 \in R_{p,m+\gamma}} \lambda_{j_3}\lambda_{p-j_3}.$$

Now since the sum in γ has $O(1)$ terms, we can repeat the Cauchy-Schwarz argument from (3.32) and obtain the desired bound.

We now treat the main case, assuming $|j_2 - j_1| > 2$, by addressing two points:

- **The range of the sum in γ.**

  Noting that $j_1 \in R_{p,m}$ implies $|j_1 - j_2| = \sqrt{2m - |p|^2}$, from (3.38) we obtain

  $$\sqrt{2m - |p|^2} - \tfrac{4}{5} \le \sqrt{2(m+\gamma) - |p|^2} \le \sqrt{2m - |p|^2} + \tfrac{4}{5}$$

  from which we find

  $$\gamma \in \left[-\tfrac{4}{5}\sqrt{2m - |p|^2} + \tfrac{8}{25}, \tfrac{4}{5}\sqrt{2m - |p|^2} + \tfrac{8}{25}\right] =: \Gamma_{m,p},$$

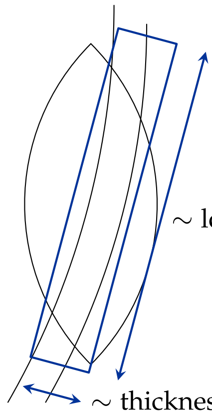
  i.e. there are $\sim \sqrt{2m - |p|^2}$ different γ's.

- **The size of $\{(q, k_1, k_3) : |\cdots| \le \frac{c}{50}\}$.**

  Once q is fixed, the worst case is if all k_1 lead to a contribution. Since k_1 must be chosen from the lens defined by q, we estimate their number by the area of the lens; this gives

  $$\#\{k_1\} \sim \left(\tfrac{1}{5}\delta^{-1/2} - |q|\right)\sqrt{\left(\tfrac{1}{5}\delta^{-1/2}\right)^2 - |q|^2}.$$

  Now to count the number of k_3, note that in the worst case the k_3 annulus will overlap the k_3 lens along its long dimension. So we can bound the number of k_3 in terms of the area of a box which contains this overlap:



We have (from "$a^2 - b^2 = (a+b)(a-b)$")

$$\begin{aligned}\sqrt{|Z|^2 + \tfrac{c}{50}\delta^{-1}} - \sqrt{|Z|^2 - \tfrac{c}{50}\delta^{-1}} &= \frac{|Z|^2 + \frac{c}{50}\delta^{-1} - (|Z|^2 - \frac{c}{50}\delta^{-1})}{\sqrt{|Z|^2 + \frac{c}{50}\delta^{-1}} + \sqrt{|Z|^2 - \frac{c}{50}\delta^{-1}}} \\ &\leq \frac{\frac{c}{25}\delta^{-1}}{|Z|}.\end{aligned}$$

Now since $|j_2 - j_1| \geq 2$, we have that the $j_2 - j_1$ part of Z is dominant, hence $|Z| \gtrsim \delta^{-1/2}|j_2 - j_1|$ and so the expression for the thickness of the annulus can be controlled by

$$\frac{\delta^{-1/2}}{|j_2 - j_1|} = \frac{\delta^{-1/2}}{\sqrt{2m - |p|^2}}.$$

Thus we get the worst-case contribution

$$\#\{k_3\} \sim \sqrt{\left(\tfrac{1}{5}\delta^{-1/2}\right)^2 - |q|^2}\,\frac{\delta^{-1/2}}{\sqrt{2m - |p|^2}}.$$

Hence

$$\begin{aligned}\#\{(q, k_1, k_3) \,:\, |\cdots| \leq \delta\} &\lesssim \sum_q (\tfrac{1}{5}\delta^{-1/2} - |q|)(\tfrac{1}{25}\delta^{-1} - |q|^2)\frac{\delta^{-1/2}}{\sqrt{2m - |p|^2}} \\ &\lesssim \delta^{-3/2}\delta^{-1/2}\sum_q \frac{1}{\sqrt{2m - |p|^2}} \\ &\lesssim \delta^{-3}\frac{1}{\sqrt{2m - |p|^2}}\end{aligned}$$

since there are $\sim \delta^{-1}$ terms in the q sum.

Putting this information into (3.36), we now have a bound of

$$\sum_p \sum_m \frac{\delta^{-3}}{\sqrt{2m-|p|^2}} \sum_{j_1 \in R_{p,m}} \lambda_{j_1} \lambda_{p-j_1} \sum_{\gamma \in \Gamma_{m,p}} \sum_{j_3 \in R_{p,m+\gamma}} \lambda_{j_3} \lambda_{p-j_3}.$$

Applying Cauchy-Schwarz to both the j_1 and γ/j_3 sums gives the bound

$$\sum_p \sum_m \frac{\delta^{-3}}{\sqrt{2m-|p|^2}} |R_{p,m}|^{\frac{1}{2}} \left(\sum_{j_1 \in R_{p,m}} (\lambda_{j_1} \lambda_{p-j_1})^2 \right)^{\frac{1}{2}} (\# j_3)^{\frac{1}{2}} \left(\sum_{\gamma \in \Gamma_{m,p}} \sum_{j_3 \in R_{p,m+\gamma}} (\lambda_{j_3} \lambda_{p-j_3})^2 \right)^{\frac{1}{2}}.$$

Now using Lemma 3.10, $|R_{p,m}| \lesssim \delta^{-\varepsilon}$ for any p, m in the range of summation. Similarly, $|R_{p,m+\gamma}| \lesssim \delta^{-\varepsilon}$, so the contribution of $(\# j_3)^{1/2}$ is $\delta^{-\varepsilon} |\Gamma_{p,m}|^{1/2} \lesssim \delta^{-\varepsilon} \sqrt{2m-|p|^2}^{1/2}$. Putting this in, we have

$$\delta^{-3-\varepsilon} \sum_p \sum_m \left(\sum_{j_1 \in R_{p,m}} (\lambda_{j_1} \lambda_{p-j_1})^2 \right)^{\frac{1}{2}} \left(\frac{1}{\sqrt{2m-|p|^2}} \sum_{\gamma \in \Gamma_{m,p}} \sum_{j_3 \in R_{p,m+\gamma}} (\lambda_{j_3} \lambda_{p-j_3})^2 \right)^{\frac{1}{2}}.$$

A final application of Cauchy-Schwarz shows that this is bounded by

$$\delta^{-3-\varepsilon} \left(\sum_p \sum_m \sum_{j_1 \in R_{p,m}} (\lambda_{j_1} \lambda_{p-j_1})^2 \right)^{\frac{1}{2}} \left(\sum_p \sum_m \frac{1}{\sqrt{2m-|p|^2}} \sum_{\gamma \in \Gamma_{m,p}} \sum_{j_3 \in R_{p,m+\gamma}} (\lambda_{j_3} \lambda_{p-j_3})^2 \right)^{\frac{1}{2}}.$$

Now in the first bracket note that, once p and j_1 are chosen, the value of $m = |j_1|^2 + |p-j_1|^2$ is determined. So that term can be written simply as

$$\left(\sum_p \sum_j (\lambda_j \lambda_{p-j})^2 \right)^{\frac{1}{2}}.$$

For the other bracket, we have

$$\sum_p \sum_m \sum_{\gamma \in \Gamma_{m,p}} \sum_{j_3 \in R_{p,m+\gamma}} \frac{(\lambda_{j_3} \lambda_{p-j_3})^2}{\sqrt{2m-|p|^2}} = \underbrace{\sum_p \sum_{j_3} \sum_m}_{\text{s.t. } |j_3-j_4| \in \sqrt{2m-|p|^2} + \left[-\frac{4}{5}, \frac{4}{5}\right]} \frac{(\lambda_{j_3} \lambda_{p-j_3})^2}{\sqrt{2m-|p|^2}}.$$

With p and j_3 fixed, we can see that there are $\lesssim |j_3 - j_4|$ values of m which satisfy the condition. We also know that $\sqrt{2m-|p|^2} \sim |j_3 - j_4|$, so this gives the bound

$$\sum_p \sum_{j_3} |j_3 - j_4| \frac{(\lambda_{j_3} \lambda_{p-j_3})^2}{|j_3 - j_4|} \lesssim \sum_p \sum_{j_3} (\lambda_{j_3} \lambda_{p-j_3})^2.$$

Thus we have

$$\begin{aligned}
\text{LHS} &\lesssim \delta^{-3-\varepsilon}\left(\sum_p\sum_j(\lambda_j\lambda_{p-j})^2\right)^{\frac{1}{2}}\left(\sum_p\sum_j(\lambda_j\lambda_{p-j})^2\right)^{\frac{1}{2}}\\
&\lesssim \sum_p\sum_j(\lambda_j\lambda_{p-j})^2\\
&\lesssim \left(\sum_j\lambda_j^2\right)^2,
\end{aligned}$$

showing that (3.14) does indeed hold. ∎

We have not been able to obtain the same result for the $n=2$ case, primarily because the sets of points involved seem more complicated. Specifically, if we view the condition $\left|A+2\delta^{1/2}B+\delta C\right|\leq\frac{c}{50}$ as defining the set of possible (k_4,k_5) once all the j_i and k_1,k_2,k_3 are fixed, we obtain a thickened ellipse (rather than the annulus we obtained in (3.37)). We have not been able to fully determine how the values of j_i and k_1,k_2,k_3 affect this ellipse, and in any case, it seems that the analysis leading to (3.38) would be much more complicated for a thickened ellipse rather than a thickened circle.

3.6 Test case: same distribution on each slab

We suppose that $a_{jk}=b_k$, i.e. the values attached to each blob are independent of the slab. We also suppose that the $b_k\geq 0$.

For the $n=3$ case, the right-hand side of (3.14) is then

$$A^4_{\delta,c}\left(\sum_j\left(\sum_{k_1+k_2=k_3+k_4}b_{k_1}b_{k_2}b_{k_3}b_{k_4}\right)^{1/2}\right)^2\sim A^4_{\delta,c}\delta^{-2}\sum_{k_1+k_2=k_3+k_4}b_{k_1}b_{k_2}b_{k_3}b_{k_4}$$

as there are $\sim\delta^{-1}$ different j's.

Starting with the equivalent form (3.18) of the left-hand side, we get

$$\begin{aligned}
&\underbrace{\sum_p\sum_m\sum_{j_1\in R_{p,m}}\sum_\gamma\sum_{j_3\in R_{p,m+\gamma}}\sum_q\sum_{k_1}\sum_{k_3}}_{\text{s.t.}\left|A+2\delta^{1/2}B+\delta C\right|\leq\frac{c}{50}}b_{k_1}b_{q-k_1}b_{k_3}b_{q-k_3}\\
=&\sum_q\sum_{k_1}\sum_{k_3}\#\left\{(p,j_1,j_3)\,:\,\left|A+2\delta^{1/2}B+\delta C\right|\leq\frac{c}{50}\right\}b_{k_1}b_{q-k_1}b_{k_3}b_{q-k_3}.
\end{aligned}$$

We now establish (3.14) in this case by proving

Theorem 3.24.

$$\#\left\{(p,j_1,j_3)\;:\;\left|A+2\delta^{1/2}B+\delta C\right|\leq\tfrac{c}{50}\right\}\lesssim\delta^{-2-\varepsilon}$$

for every choice of q,k_1,k_3.

Proof. Fix $q=\widetilde{q}$, $k_1=\widetilde{k_1}$,$k_3=\widetilde{k_3}$, and consider the example with $b_{\widetilde{k_1}}=b_{\widetilde{q}-\widetilde{k_1}}=b_{\widetilde{k_3}}=b_{\widetilde{q}-\widetilde{k_3}}=1$ and all other $b_k=0$. Since this is an example with at most four nonzero blobs per slab, we know from Theorem 3.16 that the inequality holds in this case with $A_{\delta,c}=C_\varepsilon\delta^{-\varepsilon}$. Thus

$$\underset{\text{where } k_i\in\{\widetilde{k_1},\widetilde{q}-\widetilde{k_1},\widetilde{k_3},\widetilde{q}-\widetilde{k_3}\}}{\sum_q\sum_{k_1}\sum_{k_3}}\#\left\{(p,j_1,j_3)\;:\;\left|A+2\delta^{1/2}B+\delta C\right|\leq\tfrac{c}{50}\right\}\lesssim\delta^{-2-\varepsilon}.\tag{3.39}$$

Notice that expanding out the left-hand side, we have a finite sum of the $\#\{\cdots\}$ expressions, each with a different choice of q,k_1,k_3. If any term had $\#\{\cdots\}\gtrsim\delta^{-2-\varepsilon}$ then (3.39) could not hold, so they must all be $\lesssim\delta^{-2-\varepsilon}$. In particular, the term with $q=\widetilde{q}$, $k_1=\widetilde{k_1}$, $k_3=\widetilde{k_3}$ has

$$\#\left\{(p,j_1,j_3)\;:\;\left|A+2\delta^{1/2}B+\delta C\right|\leq\tfrac{c}{50}\right\}\lesssim\delta^{-2-\varepsilon}$$

and since the choice of $\widetilde{q},\widetilde{k_1},\widetilde{k_3}$ was arbitrary, this must hold for any such choice. ■

For the $n=2$ case, we find that (3.15) becomes

$$\begin{aligned}\sum_q\sum_{k_1,k_2}\sum_{k_4,k_5}\#\left\{(p,j_1,j_2,j_4,j_5)\;:\;\left|A+2\delta^{1/2}B+\delta C\right|\leq\tfrac{c}{50}\right\}b_{k_1}b_{k_2}b_{q-k_1-k_2}b_{k_4}b_{k_5}b_{q-k_4-k_5}\\ \lesssim A^6_{\delta,c}\delta^{-3/2}\sum_{k_1+k_2+k_3=k_4+k_5+k_6}b_{k_1}b_{k_2}b_{k_3}b_{k_4}b_{k_5}b_{k_6},\end{aligned}$$

but the argument above is easily adapted to this case; we can establish

$$\#\left\{(p,j_1,j_2,j_4,j_5)\;:\;\left|A+2\delta^{1/2}B+\delta C\right|\leq\tfrac{c}{50}\right\}\lesssim\delta^{-3/2-\varepsilon}$$

as in the proof of Theorem 3.24, by making use of the $n=2$ result for fixed finite blobs, Theorem 3.18.

APPENDIX

A Maple calculations

A.1 Counting the number of patterns

A.1.1 Large patterns

The following code checks each of the 64 possible `triples` with $\omega_1 \in Q_1$, $\omega_2 \in Q_2$ and $\omega_3 \in Q_3$, and selects only those where $\omega_1 \wedge \omega_2 \wedge \omega_3$ can be large enough.

```
Sq:=(i,j)->[ [i-1,j-1],[i-1,j],[i,j],[i,j-1] ]:
Atri:=proc(x,y,z)
  abs(1/2*LinearAlgebra[Determinant](Matrix([[x[],1],[y[],1],[z[],1]])));
end proc:
wedgerange:= proc(b1,b2,b3)
  local ws,s1:=Sq(b1[]),s2:=Sq(b2[]),s3:=Sq(b3[]);
  ws:=[seq(seq(seq( Atri(w1,w2,w3), w1 in s1 ), w2 in s2), w3 in s3)];
  return [min(ws),max(ws)];
end proc:
Q[1]:={seq(seq([i,j],i=1..2),j=1..2)}:
Q[2]:={seq(seq([i,j],i=1..2),j=3..4)}:
Q[3]:={seq(seq([i,j],i=3..4),j=1..2)}:
triples:={seq(seq(seq([w1,w2,w3],w3 in Q[3]),w2 in Q[2]), w1 in Q[1])}:
goodtriples:=select(w->is(max(wedgerange(w[]))>4),triples):
nops(goodtriples);
```

This shows that there are 39 large patterns among the 64 candidates.

A.1.2 Fine patterns

Here we consider the 4096 possible `triples` of squares with $\omega_1 \in Q_1$, $\omega_2 \in Q_1$ and $\omega_3 \in Q_3$, and selects only those where $\omega_1 \wedge \omega_2 \wedge \omega_3$ can be large enough.

```
Sq:=(i,j)->[ [i-1,j-1],[i-1,j],[i,j],[i,j-1] ]/2:
Atri:=proc(x,y,z)
  abs(1/2*LinearAlgebra[Determinant](Matrix([[x[],1],[y[],1],[z[],1]])));
```

```
end proc:
wedgerange:= proc(b1,b2,b3)
  local ws,s1:=Sq(b1[]),s2:=Sq(b2[]),s3:=Sq(b3[]);
  ws:=[seq(seq(seq( Atri(w1,w2,w3), w1 in s1 ), w2 in s2), w3 in s3)];
  return [min(ws),max(ws)];
end proc:
Q[1]:={seq(seq([i,j],i=1..4),j=1..4)}:
Q[4]:={seq(seq([i,j],i=5..8),j=5..8)}:
triples:={seq(seq(seq([w1,w2,w3],w3 in Q[4]),w2 in Q[1]), w1 in Q[1])}:
goodtriples:=select(w->is(max(wedgerange(w[]))>4),triples):
nops(goodtriples);
```

We find that there are 154 fine patterns among these candidates.

A.2 Counting combinations satisfying the ABC condition

```
with(plots):with(plottools):with(LinearAlgebra):
N:=12^2;
delta:=N^(-1);
sqrtN:=sqrt(N);
beta[1]:=5;
beta[2]:=5;

injdisc := proc(j)
  option remember;
  return is(j[1]^2+j[2]^2 <= ((1/beta[1])*delta^(-1/2))^2 ):
end proc:
w:=(j,k)-> <s(j,k)[1],s(j,k)[2],s(j,k).s(j,k)/2>;
s:=(j,k)-> beta[1]*sqrt(delta)*j+beta[2]*delta*k;
testvals:=proc(A,B,C):
  return evalf(abs((beta[1]^2)*A+(2*beta[1]*beta[2]*delta^(1/2))*B
                                                    +(beta[2]^2*delta)*C));
end proc:

js:=select(injdisc,{seq(seq(<j1,j2>,j1=-sqrtN..sqrtN),j2=-sqrtN..sqrtN)}):
nops(js); # possible choices of j to check
jcombos:=select( j->is(injdisc(j[4])),
         {seq(seq(seq([j1,j2,j3,j1+j2-j3],j1 in js),j2 in js),j3 in js)}):
nops(jcombos); # combinations j1+j2=j3+j4
```

This shows that

- there are 21 possible choices of j to check;
- using these, there are 4381 combinations $j_1 + j_2 = j_3 + j_4$;
- since the set of possible k's is the same, there are also 4381 combinations $k_1 + k_2 = k_3 + k_4$.

We now sort the combinations $j_1 + j_2 = j_3 + j_4$ according to the value of A:

```
jcomboswithA:=table():
jcombos:=table():
count:=0:
for j1 in js do:
for j2 in js do:
for j3 in js do:
  j4:=j1+j2-j3;
  if(injdisc(j4)) then:
    count:=count+1;
    jcombos[count]:=[j1,j2,j3,j4];
    A:=j1.j1+j2.j2-j3.j3-j4.j4;
    if(jcomboswithA[A]::table) then:
      jcomboswithA[A][count]:=count;
    else:
      jcomboswithA[A]:=table([ count=count ]);
    end if:
  end if:
end do:
end do:
end do:
AvsCount:=sort(map( p->[op(1,p),nops([entries(op(2,p))])],
                    [entries(jcomboswithA,pairs)]));
```

This returns the following data:

```
[[-10, 8], [-8, 108], [-6, 208], [-4, 540], [-2, 572], [0, 1509],
    [2, 572], [4, 540], [6, 208], [8, 108], [10, 8]]
```

Finally, we check each choice of j_i and k_i satisfying the two equations in (3.17) to see if the condition $\left|A + 2\delta^{1/2}B + \delta C\right| \leq \frac{4}{25}$ is satisfied:

```
hits:=table():
counter:=0:

for Aval in sort({indices(jcomboswithA,nolist)}) do:
  print(Aval);
  for jc in [entries(jcomboswithA[Aval],nolist)] do:
  j1,j2,j3,j4:=jcombos[jc][]:
  for kcombo in [entries(jcombos,nolist)] do:
    k1,k2,k3,k4:=kcombo[];
    B:=j1.k1+j2.k2-j3.k3-j4.k4;
    C:=k1.k1+k2.k2-k3.k3-k4.k4;
    if((Aval=0 and B=0) or testvals(Aval,B,C)<=4) then:
      # when A=0, B=0, C is always small enough to satisfy the condition
      counter:=counter+1;
      hits[counter]:=[Aval,B,C,jcombos[jc],kcombo];
    end if:
  end do:
  end do:
end do:

nops(select(A->is(A=0),map(h->op(1,h), [entries(hits,nolist)])));
nops(select(A->is(A<>0),map(h->op(1,h), [entries(hits,nolist)])));
```

This took around 9 hours to run, and reported that of the $4381^2 \approx 19$ million choices of j_i and k_i,

- 1 669 521 ($\approx$ 8.70%) satisfy ABC condition, with $A = 0$;
- 14 656 ($\approx$ 0.08%) satisfy ABC condition, with $A \neq 0$.

A.3 Counting combinations satisfying the ABC condition

The following code sets up the problem with $\delta = 1/(30^2)$, $\beta_1 = \beta_2 = 10$.

```
with(LinearAlgebra):
injdisc := proc(j):
  return is(j[1]^2+j[2]^2 <= ((1/beta[1])*delta^(-1/2))^2 ):
end proc:
inkdisc := proc(k):
  return is(k[1]^2+k[2]^2 <= ((1/beta[2])*delta^(-1/2))^2 ):
end proc:
N:=30^2;
delta:=N^(-1);
sqrtN:=sqrt(N);
beta[1]:=10;
beta[2]:=10;
w:=(j,k)-> <s(j,k)[1],s(j,k)[2],s(j,k).s(j,k)/2>;
s:=(j,k)-> beta[1]*sqrt(delta)*j+beta[2]*delta*k;
makeABC:=proc(jcombo,kcombo)
  option remember;
  j1,j2,j3,j4:=jcombo[]:
  k1,k2,k3,k4:=kcombo[]:
  A:=j1.j1+j2.j2-j3.j3-j4.j4;
  B:=j1.k1+j2.k2-j3.k3-j4.k4;
  C:=k1.k1+k2.k2-k3.k3-k4.k4;
  return(A,B,C);
end proc:
testvals:=proc(A,B,C)
  option remember;
  return evalf(abs((beta[1]^2)*A+(2*beta[1]*beta[2]*delta^(1/2))*B
                                            +(beta[2]^2*delta)*C));
end proc:

js:=Array():
js:={<0,0>}:
for j1 from -sqrtN to sqrtN do:
for j2 from -sqrtN to sqrtN do:
  if(injdisc([j1,j2]) ) then:
    js := {js[],<j1,j2>};
  end if:
end do:
end do:
nops(js),"possible␣choices␣of␣j␣to␣check";
ks:=Array():
```

```
ks:={<0,0>}:
for k1 from -sqrtN to sqrtN do:
for k2 from -sqrtN to sqrtN do:
  if(inkdisc([k1,k2]) ) then:
    ks := {ks[],<k1,k2>};
  end if:
end do:
end do:
nops(ks),"possible␣choices␣of␣k␣to␣check";
tjs:=table():
tjsi:=table():
counter:=1:
for j1 from -sqrtN to sqrtN do:
for j2 from -sqrtN to sqrtN do:
  if(injdisc([j1,j2]) ) then:
    tjs[counter] := <j1,j2>;
    tjsi[[j1,j2]] := counter;
    counter:=counter+1;
  end if:
end do:
end do:
numtjs:=nops([entries(tjs,nolist)]);
# since the set of js and ks is the same, just cheat in making the tks...
tks:=eval(tjs);
tksi:=eval(tjsi);
numtks:=nops([entries(tks,nolist)]);
```

Our goal is to compare the value of the sum under the two regimes: "$k_j = k^*$ for each j", and "each k_j arbitrary". One can easily check that if there are any equalities holding between the j_i, then the contribution is the same in either regime; so to establish if taking $k_j = k^*$ gives the maximum value, it suffices to check the contributions when all j_i are distinct. We now find those:

```
# find combos such that j1+j2=j3+j4, in enumerated form
djcombos:=table():
djcombos[1]:=[0,0,0,0]:
combocount:=0:
for j1 from 1 to numtjs do:
for j2 from 1 to numtjs do:
for j3 from 1 to numtjs do:
  j4v:=tjs[j1]+tjs[j2]-tjs[j3]:
  if(injdisc(j4v)) then:
    j4:=tjsi[[j4v[1],j4v[2]]];
    if(is([j1,j2,j4,j3] in {entries(djcombos,nolist)}
          or [j2,j1,j3,j4] in {entries(djcombos,nolist)}
          or [j2,j1,j4,j3] in {entries(djcombos,nolist)}
          or [j3,j4,j1,j2] in {entries(djcombos,nolist)}
          or [j4,j3,j1,j2] in {entries(djcombos,nolist)}
          or [j4,j3,j2,j1] in {entries(djcombos,nolist)}
          or [j3,j4,j1,j2] in {entries(djcombos,nolist)} )) then: next:
    end if:
    combocount:=combocount+1;
    djcombos[combocount] := [j1,j2,j3,j4];
```

```
  end if:
end do:
end do:
end do:
combs:=[entries(djcombos,nolist)]:
nops(combs);
dcombs:=select(c->is(nops({c[]})=4),combs):
nops(dcombs);
```

The following code lets us vary the positions of the blobs; `Q(0)` gives the value of the sum when $k_j = k^*$ for all j, while `Q(numtjs)` gives the value when the k_j are randomly positioned. The procedure `trial()` is used to randomly select the blob positions, and the values to attach to each blob, before computing the sum in the two regimes.

```
heatk:= (tj,t) -> `if`(t<tj,1,tkj[tj]);
Q:=proc(t):
hits:=0:
sumval:=0:
for tjcombo in dcombs do:
  tj1,tj2,tj3,tj4:=tjcombo[]:
  jcombo:=[tjs[tj1],tjs[tj2],tjs[tj3],tjs[tj4]]:
  j:=jcombo:
  kcombo:=[tks[heatk(tj1,t)],tks[heatk(tj2,t)],
                 tks[heatk(tj3,t)],tks[heatk(tj4,t)]]:
  k:=kcombo:
  if((Norm(kcombo[1]+kcombo[2]-kcombo[3]-kcombo[4]))=0) then:
    A,B,C:=makeABC(jcombo,kcombo):
    if(testvals(A,B,C)<=4) then:
      hits:=hits+1;
      sumval:=sumval+a[jcombo[1]]*a[jcombo[2]]*a[jcombo[3]]*a[jcombo[4]];
    end if:
  end if:
end do:
return [hits,sumval];
end proc:
trial := proc()
  global tkj,a;
  TKJ:=[seq(rand(1..numtks)(),i=1..numtjs)];
  vals:=[seq(rand(0..1)(),i=1..numtjs)];
  tkj:=table():
  for t from 1 to nops(TKJ) do:
    tkj[t] := TKJ[t];
  end do:
  a:=table():
  for t from 1 to numtjs do:
    a[tjs[t]] := vals[t];
  end do:
  endresult:=Q(numtjs);
  if(endresult=[0,0]) then:
    return [TKJ,vals,"n/a",[0,0]];
  else
    return [TKJ,vals,Q(0),endresult];
  end if:
```

```
end proc:
```

We now run this experiment 100 times:

```
numtrials:=100;
numconfirmations:=0:
for i from 1 to numtrials do:
  res:=trial();
  if(res[4]=[0,0] or (res[3,2]>=res[4,2] and res[3,1]>=res[4,1])) then:
    numconfirmations:=numconfirmations+1;
  end if:
  print(trial());
end do:
numconfirmations;
```

We found that the conjecture was verified in every case. When `Q(numtjs)` was nonzero, it generally had 2 or 4 terms, while the $k_j = k^*$ case would usually have 342 terms — significantly more.

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